Optimal Marketplace Design

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Abstract

In financial markets as well as online marketplaces, each user can be a buyer or a seller depending on the market conditions and their endowments. Here, I consider the problem of designing a marketplace for such a market with a divisible good to maximize profit. I first focus on Dominant-Strategy Implementable mechanisms and invoke the revelation principle. I show that the designer's profit is the expected virtual surplus. Then, I describe the optimal allocation through an algorithm. The algorithm ranks agents according to their virtual values and costs and allows trade between two agents if one's value is greater than the other's cost. The volume of trade is determined by their endowments. After finding the optimal Dominant Strategy Implementable mechanism, I argue that this mechanism is in fact optimal within the class of Bayesian Implementable mechanisms as well. Finally, I consider an extension where the marketplace itself can own some endowments and illustrate the type of inefficiency this can lead.

1 Introduction

In this paper, I analyze optimal marketplace design problem in an economy with a single, divisible good, and agents who can be both buyers or sellers, depending on the market conditions.

Stock exchanges are early examples of markets where each participant can buy more stocks or sell some. However, online marketplaces that emerged more recently also have this feature. These marketplaces that bring buyers and sellers together to make profit out of their trades are everywhere. Uber, Airbnb and Amazon are some of the prominent examples that became parts of daily life. They allow users to be on either demand or supply side of the market easily. This introduces an interesting problem for the marketplace: It needs to decide which agent it wants as buyers and sellers before it can choose an optimal payment structure.

I study the problem of designing a marketplace to maximize its profit in an environment with a single good. There are finitely many agents. Each agent has some endowments of the good, and demand for 1 unit of the good. I assume that the good is divisible and endowments are between 0 and 1. Agents' marginal valuations for the good is their private information. However, the designer knows the distribution from which these valuations are drawn.

I first focus on Dominant Strategy Implementable Mechanisms. (Later I show that this is without loss of profit.) By revelation principle for dominant strategy equilibrium, I focus on direct, Dominant Strategy Incentive Compatible Mechanisms. I show that the profit of the marketplace can be written as the expected sum of the virtual values and virtual costs of agents, depending on whether they are buyers or sellers.

Given this, it becomes easy to characterize the optimal allocation through an algorithm. I rank all agents according to their virtual values and virtual costs. I compare the highest virtual value, $\mathcal{V}_i$ to the lowest virtual cost, $\mathcal{C}_j$ and shut down the mechanism if the value is less than the cost: This means even the most profitable trade is not profitable for the marketplace. The mechanism allows trade between these two agents -at the maximum possible volume- if $\mathcal{V}_i > \mathcal{C}_j$. Volume of trade is determined by agents' endowment levels: They trade at a quantity such that either the agent with the highest virtual value has no more demand or the agent with the lowest virtual cost has no more endowment. If one's demand is satisfied or the supply is exhausted, she is removed from the process with her current allocation. Next, the algorithm moves on to the highest virtual value and the lowest virtual cost among the remaining agents and makes the same comparison as above. It stops when there is no more virtual value greater than virtual cost among the remaining agents. The transfers in the optimal marketplace are pinned down by an interplay of incentive compatibility, individual rationality and profit-maximization.

Next, I show that the restriction to Dominant Strategy Implementable Mechanisms is without loss. I do this in two steps. First, I assume that the good is indivisible and show that in this case, the environment studied here is within the scope of the Bayesian and Dominant Strategy equivalence result of Gershkov et al. (2013). Then, under the assumption that agents endowments are rational numbers, I show that the equivalence result from Gershkov et al. (2013) extends to this setup. The restriction to rational endowments is for technical reasons and can be extended. However, in any economic environment we are familiar with, endowments are rational numbers. Thus, the marketplace does not lose profit by focusing on Dominant Strategy Implementation and in fact gains the additional robustness provided by dominant strategies.

Finally, I consider an extension where the marketplace itself owns some endowment and can become a seller. I argue that a simple modification of the algorithm described above solves this problem. In this case, the marketplace ranks itself among agents with its virtual cost equal to its actual cost. Of course, the inefficiency this implies is clear. The marketplace rather sells itself before letting another agent with same or even slightly lower valuation sell.

A recent report by US Congress (Congress Majority Staff, 2020) has argued that Amazon's practice of producing products similar to 'best selling' products in many categories and selling them under Amazon Basics brand is harmful for the economy. Furthermore, the report claims that Amazon starts stocking 'best selling' products itself, which causes losses for the sellers who created third-party stores on Amazon Marketplace and arguably made the product popular in the first place. The extension with the marketplace as a seller might seem to justify the concern. However, it is important to note that the inefficiency I highlight here is about how the 'producer's surplus' is shared; Amazon's low-priced products might still be improving the consumer welfare.

1.1 The Simple Economics of Optimal Marketplaces

In this section, I illustrate the revenue-maximizing allocation rule using an analysis similar to Bulow and Roberts (1989). For simplicity, suppose each agent has the endowment of 0.5 unit so that each of them can buy or sell up to 0.5 unit of the good. Suppose agent i valuation θ_i is drawn from the distribution F_i. We define virtual values and costs:

$$B_i(\theta_i) = \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta)} \text{ and } S_i(\theta_i) = \theta_i + \frac{F_i(\theta_i)}{f_i(\theta)}.$$

As I explained, the objective function can be written as the expected sum of each agent's virtual value or virtual cost, depending on whether the agent is ultimately assigned the role of a buyer or a seller.

I will now use virtual values and costs to define the virtual (inverse) demand and supply functions. we rank the agents' virtual values highest to lowest and denote them $b_1, b_2, \ldots, b_n$. Similarly, we rank them according to the virtual costs but from the lowest to the highest this time and denote those numbers as $s_1, s_2, \ldots, s_n$. We define the virtual demand D and supply S as

$$D(q) = b_i \text{ if } \frac{i-1}{2} < q \leq \frac{i}{2} \text{ and } S(q) = s_i \text{ if } \frac{i-1}{2} < q \leq \frac{i}{2}.$$

This is in line with the way inverse demand and supply functions are taught as nonincreasing and nondecreasing functions of quantity in introductory level courses. However, here they are step functions with steps of length 0.5 since there are finitely many agents with 0.5 unit of demand and supply each.

I can now express a simple version of the Theorem 2.1 using the virtual demand and supply functions.

Corollary 1.1. *The allocation rule associated with the mechanism that maximizes the profit under incentive compatibility and individual rationality is as follows:*

If $B(0.5) \leq S(0.5)$*, no trade takes place.*

If $B(0.5) > S(0.5)$*, then the total volume of trade is given by*

$$q^* = \max_{q \in [0, 0.5n]} \{q | B(q) > S(q)\}.$$

Moreover, each agent with a virtual value greater than $B(q^*)$ *buys* 0.5 *unit; each agent with a virtual cost less than* $S(q^*)$ *sells* 0.5 *unit; the rest do not trade.*

We will now study a concrete example to see how the allocation rule would work. Suppose there are 4 agents. For the sake of simplicity, assume that each agent i's valuation θ_i is drawn from the uniform distribution on $[0, 1]$. In this case, virtual values and costs are given by $2\theta_i - 1$ and $2\theta_i$ respectively. Suppose the principal receives the reports $0.2, 0.4, 0.6, 0.8$. This would translate to the virtual values $-0.6, -0.2, 0.2, 0.6$ and virtual costs $0.4, 0.8, 1.2, 1.6$. If we plot the supply and demand, we would get Figure 1 (left). The corollary above implies that in the profit-maximizing mechanism, total quantity traded is 0.5 and agent whose value is 0.8 buys 0.5 unit, agent whose value is 0.2 sells 0.5 and the other agents do not participate in any trade. For instance, if the agent whose valuation is 0.8 valued the good at 0.6, then the corresponding plot is provided in Figure 1 (right). Since the highest value of D is lower than the lowest value of S, in the optimal mechanism, there would not be any trade with these realizations.

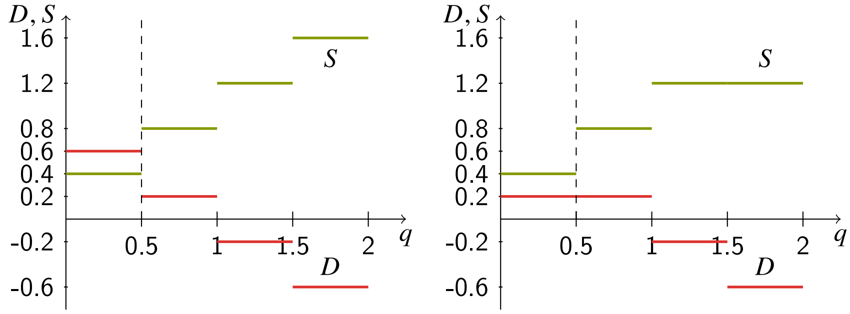

Figure 1: Demand and Supply curves given realization $\{0.2, 0.4, 0.6, 0.8\}$ (left) and $\{0.2, 0.4, 0.6, 0.6\}$ (right). In the right, D is always below S, hence the corollary tells that there would not be any trade in this instance. However, in the left, D is above S up to $q = 0.5$. Hence, according to the corollary, exactly 0.5 unit of good should be traded in the platform.

1.2 Literature Review

Myerson and Satterthwaite (1983) study the problem of designing a bilateral trade mechanism for one buyer and one seller, with a single, indivisible good, either to maximize the total welfare or the profit of the mechanism. However, there the roles as buyer and seller are exogenously determined. By contrast, here the good is divisible, agents have arbitrary endowments and the mechanism has to decide whether to make an agent a buyer or a seller, or not trade with them.

Lu and Robert (2001) analyze a closely related problem to mine. They also focus on a setup with a divisible good, arbitrary endowments but unlike this paper, they assume the valuations are drawn from the same distribution. Moreover, they directly attack the Bayesian Implementation problem to maximize a mix of the welfare and the profit. While they are able to provide conditions that the optimal allocations need to satisfy and show that an optimal allocation exists, the solution is quite complicated and hard to implement. Moreover, the solution may not be dominant-strategy implementable. By contrast, here I provide an explicit, closed-form solution through the algorithm I describe, the mechanism is dominant-strategy incentive compatible, and I show that it doesn't lose any profit compared to optimal Bayesian mechanism.

Finally, Idem (2021) first extends the results from this paper to an environment with a continuum of agents. Then, it considers a market choice game where the designer announces the details of the marketplace and the agents choose between the marketplace and a decentralized market.

It shows that in the unique equilibrium, both markets are active; the agents with 'intermediate' values join the decentralized market while the rest of the agents join the centralized market.

2 Monagora

I call this model monagora to emphasize the fact that in this part of the paper, I restrict all trade to the centralized marketplace; hence there is a unique market or monagora.

2.1 Setup

- Agents: $n > 1$ agents; $N = \{1, \ldots, n\}$.
- Types: Each agent i has a value for a single-unit of a single good, $\theta_i \in \Theta_i$ which is private information. (Let $\Theta = \prod_{i=1}^{n} \Theta_i$ and $\theta \in \Theta$). I assume they have quasi-linear preferences and that each agent has a unit-demand for the good.
- Endowments: Each agent i has $e_i \in [0,1]$ units of good which the designer and the agents take as given. All endowments are common knowledge among the agents and a mechanism designer.
- A mechanism designer wants to design a mechanism to maximize its profit.

By revelation principle, I focus on direct mechanisms that allocates $q_i : \theta \times [0,1]^n \to \mathcal{R}$ units of good to each agent i and asks her to pay $t_i : \theta \times [0,1]^n \to \mathcal{R}$. Hence, the utility of the agent i from the mechanism with the valuation θ_i and endowment e_i is

$$u_i(\theta, e) = \theta_i \min\{1, q_i(\theta_i, \theta_{-i}, e) + e_i\} - t_i(\theta_i, \theta_{-i}, e) - \theta_i e_i.$$

$$\max_{(q_i,t_i)_{i\in N}} \int_\Theta \sum_{i=1}^n t_i(\theta,e)f(\theta)d\theta$$

s. t.

$$\begin{aligned}
&\text{(IC)} && \theta_i \min\{1, q_i(\theta_i,\theta_{-i},e)+e_i\} - t_i(\theta_i,\theta_{-i},e)\\
& && \geq \theta_i \min\{1, q_i(\theta_i',\theta_{-i},e)+e_i\} - t_i(\theta_i',\theta_{-i},e)\\
&\text{(IR)} && \theta_i \min\{1, q_i(\theta_i,\theta_{-i},e)+e_i\} - t_i(\theta_i,\theta_{-i},e) \geq e_i\theta_i\\
&\text{(Individual Feasibility)} && q_i(\theta,e) \geq -e_i\\
&\text{(Aggregate Feasability)} && 0 = \textstyle\sum_{i=1}^n q_i(\theta,e)
\end{aligned}$$

2.2 Simplifying The Designer's Problem

We first develop a series of lemmata that help us state the maximization problem above as a concave program.

Lemma 2.1 (Monotonicity)**.** *Suppose $(q_i,t_i)_{i\in N}$ is a direct, IC mechanism. Then,*

1. *If $q_i(\theta,e)+e_i<1$ for some $\theta\in\Theta$, $e\in[0,1]^n$ $i\in N$, then $q_i(\theta_i,\theta_{-i},e)$ is increasing in θ_i at (θ,e).*

2. *If $q_i(\theta,e)+e_i\geq 1$ for some $\theta\in\Theta$, $e\in[0,1]^n$ $i\in N$, then $q_i(\theta_i',\theta_{-i},e)+e_i\geq 1$ for each $\theta_i'\geq\theta_i$.*

The proof is standard, except for taking care of the capacities so it can be found in the Appendix A.

The next lemma presents the derivative of the utility of an agent in an IC mechanism.

Lemma 2.2 (Envelope Condition)**.** *If $(q_i,t_i)_{i\in N}$ is a direct, IC mechanism, then for each $\theta\in\Theta$*

$$\frac{\partial u(\theta,e)}{\partial\theta_i} = \begin{cases} q_i(\theta,e), & \textit{if } q_i(\theta,e)+e_i<1,\\ 1-e_i, & \textit{otherwise.}\end{cases}$$

Again, the proof is similar to standard arguments and can be found in Appendix B.

The next lemma gives the representation of the utility of each type as the integral of the allocation rule, using the previous lemma.

Lemma 2.3 (Payoff Equivalence). *If $(q_i, t_i)_{i \in N}$ is a direct, IC mechanism, then*

$$u_i(\theta, e) = u_i(\underline{\theta_i}, \theta_{-i}, e) + \int_{\underline{\theta_i}}^{\min\{\theta_i, \theta_i^*\}} q_i(x, \theta_{-i}, e)dx + (\theta_i - \min\{\theta_i, \theta_i^*\})(1 - e_i),$$

for each $\theta \in \Theta$ where θ_i^* is such that $q_i(\theta_i, e) + e_i = 1$ if such a solution exists, $\theta_i^* = \theta_i$ otherwise.

Proof. Since $u_i(\theta, e)$ is convex in θ_i restricted to regions where $q_i(\theta, e) + e_i > 1$ and $q_i(\theta, e) + e_i \leq 1$ separately, it is absolutely continuous in θ_i. Then, it is the integral of its derivative. □

Notation: For any direct mechanism $(q_i, t_i)_{i \in N}$, let

$$q_i^*(\theta, e) = \begin{cases} q_i(\theta, e), & \text{if } q_i(\theta, e) + e_i < 1, \\ 1 - e_i, & \text{otherwise.} \end{cases}$$

Note that for a direct, IC mechanism, $q_i^*(\theta_i, e)$ is also weakly increasing.

Next, we pin down the transfer rule in an IC mechanism.

Lemma 2.4 (Revenue Equivalence). *If $(q_i, t_i)_{i \in N}$ is a direct, IC mechanism, then*

$$t_i(\theta, e) \quad = -u_i(\underline{\theta_i}, \theta_{-i}, e) + \theta_i q_i^*(\theta, e) - \int_{\underline{\theta_i}}^{\theta_i} q_i^*(x, \theta_{-i}, e)dx,$$

for each $\theta \in \Theta$.

Proof. From the definition of $u_i(\theta, e)$ and the previous lemma. □

Now we provide a sufficient condition for incentive compatibility of a mechanism.

Proposition 2.1. *Let $(q_i, t_i)_{i \in N}$ be a direct mechanism. The mechanism is incentive compatible if and only if,*

1. *If $q_i(\theta, e) + e_i < 1$, then $q_i(\theta, e)$ is increasing in θ_i at (θ, e);*
2. *If $q_i(\theta, e) + e_i \geq 1$, then $q_i(\theta_i', \theta_{-i}, e) + e_i \geq 1$ for each $\theta_i' \geq \theta_i$;*
3. *$t_i(\theta_i, \theta_{-i}, e) = -u_i(\underline{\theta_i}, \theta_{-i}, e) + \theta_i q_i^*(\theta, e) - \int_{\underline{\theta_i}}^{\theta_i} q_i^*(x, \theta_{-i}, e)dx.$*

Proof can be found in Appendix C.

The next proposition provides the characterization of the IR mechanisms by establishing the types with the lowest utilities. The reason this is an issue in this model, unlike in the auction theory is that in an auction, the lowest allocation an agent could receive is 0. Hence, the utility is always increasing in agent's type, as can be seen from the envelope condition. However, here, an agent with a relatively low type can be a seller, which means he would get a negative allocation. Therefore, the utility of the lowest type is not the lowest utility in this case, which can again by seen from the envelope condition.

Proposition 2.2. *Let $(q_i, t_i)_{i \in N}$ be a direct IC mechanism. Then, it is IR if and only if for each $e \in [0,1]^n$, $\theta_{-i} \in \Theta_i$, for each agent $i \in N$,*

$$\theta_i^* q_i^*(\theta_i^*, \theta_{-i}, e) \geq t_i(\theta_i^*, \theta_{-i}, e),$$

where θ_i^* is defined as

1. $\theta_i^* = 0$ *if* $q_i^*(\underline{\theta_i}, \theta_{-i}, e) \geq 0$,

2. $\theta_i^* = \overline{\theta_i}$ *if* $q_i^*(\overline{\theta_i}, \theta_{-i}, e) < 0$,

3. *the solution to* $q_i^*(\theta_i^*, \theta_{-i}, e) = 0$ *if such a type exists,*

4. θ_i *such that for each* $\theta_i' < \theta_i$, $q_i^*(\theta_i', \theta_{-i}, e) < 0$ *and for each* $\theta_i' > \theta_i$, $q_i^*(\theta_i', \theta_{-i}, e) > 0$.

Proof. Case 1: Suppose $q_i^*(\underline{\theta_i}, \theta_{-i}, e) \geq 0$. Then, by Lemma 2.3, incentive compatibility of a mechanism implies that the associated ex-post utilities $u_i(\theta, e)$ are increasing in θ_i. Hence, if $u_i(\underline{\theta_i}, \theta_{-i}, e) \geq 0$, we have $u_i(\theta_i, \theta_{-i}, e) \geq 0$ for each $\theta_i \in \Theta_i$. Of course, $u_i(\underline{\theta_i}, \theta_{-i}, e) \geq 0$ means $\underline{\theta_i} q_i^*(\underline{\theta_i}, \theta_{-i}, e) \geq t_i(\underline{\theta_i}, \theta_{-i}, e)$

Case 2: Suppose $q_i^*(\overline{\theta_i}, \theta_{-i}, e) < 0$. Then, by Lemma 2.3, $u_i(\theta_i, \theta_{-i}, e)$ are decreasing and hence, $u_i(\overline{\theta_i}, \theta_{-i}, e)$ is the lowest payoff. Hence, if it is nonnegative, all other types' payoffs are nonnegative as above.

Cases 3 and 4: Suppose θ_i^* is defined as in the Case 3 or Case 4. Then, by Lemma 2.3, $u_i(\theta_i, \theta_{-i}, e)$ is decreasing up to θ_i^* and increasing after that point. Hence, type θ_i^* has the lowest payoff. So, if $u_i(\theta_i^*, \theta_{-i}, e) \geq 0$, each type's IR condition must also hold.

□

Lemma 2.5. *If an IC and IR mechanism maximizes the expected revenue of the designer, then for each $e \in [0,1]^n$, for each agent $i \in N$,*

$$t_i(\theta_i^*, \theta_{-i}, e) = \theta_i^* q_i(\theta_i^*, \theta_{-i}, e)$$

where θ_i^* is defined as

1. $\theta_i^* = 0$ *if* $q_i^*(\underline{\theta_i}, \theta_{-i}, e) \geq 0$,
2. $\theta_i^* = \overline{\theta_i}$ *if* $q_i^*(\overline{\theta_i}, \theta_{-i}, e) < 0$,
3. *the solution to* $q_i^*(\theta_i^*, \theta_{-i}, e) = 0$ *if such a type exists.*

Proof. The previous proposition shows that IC and IR mechanisms must have $\theta_i^* q_i(\theta_i^*, \theta_{-i}, e)$ greater than $t_i(\theta_i^*, \theta_{-i}, e)$. However, if $\theta_i^* q_i(\theta_i^*, \theta_{-i}, e) > t_i(\theta_i^*, \theta_{-i}, e)$, then the seller can increase the expected revenue by increasing $t_i(\underline{\theta_i}, \theta_{-i}, e)$ and keeping the allocation rule the same. This would increase all types' payments and the revenue strictly, contradicting revenue maximization.

□

Using the condition about θ_i^* from Lemma 2.5 and the previous lemmata, we have

$$\begin{aligned}
\theta_i^* q_i^*(\theta_i^*, \theta_{-i}, e) \quad &= t_i(\theta_i^*, \theta_{-i}, e) \\
= \quad & -u_i(\underline{\theta_i}, \theta_{-i}, e) + \theta_i^* q_i^*(\theta_i^*, \theta_{-i}, e) - \int_{\underline{\theta_i}}^{\theta_i^*} q_i^*(x, \theta_{-i}, e)dx \\
\iff \quad & u_i(\underline{\theta_i}, \theta_{-i}, e) = -\int_{\underline{\theta_i}}^{\theta_i^*} q_i^*(x, \theta_{-i}, e)dx \\
\iff \quad & t_i(\theta_i, e) = \int_{\underline{\theta_i}}^{\theta_i^*} q_i^*(x, \theta_{-i}, e)dx + \theta_i q_i^*(\theta_i, \theta_{-i}, e) - \int_{\underline{\theta_i}}^{\theta_i} q_i^*(x, \theta_{-i},, e)dx
\end{aligned}$$

Now we are ready to show that the allocation rule in a revenue-maximizing mechanism is not 'wasteful'.

Proposition 2.3. *Let $(q_i, t_i)_{i \in N}$ be a direct mechanism that maximizes the revenue of the designer. Then, for each $\theta \in \theta$, $e \in [0,1]^n$ for each $i \in N$, $q_i(\theta, e) \leq 1 - e_i$.*

Proof. First, suppose that in the optimal mechanism, there exists θ, e and j such that $q_j(\theta, e) > 1 - e_j$. Notice that decreasing the allocation to $1 - e_j$ has no effect on the agent's payoff. Hence, it doesn't effect any IC or IR constraints.

Next, let us examine the transfer rule in a direct, IC mechanism:

$$t_i(\theta_i, \theta_{-i}, e) = \int_{\underline{\theta_i}}^{\theta_i^*} q_i^*(x, \theta_{-i}, e)dx + \theta_i q_i^*(\theta, e) - \int_{\underline{\theta_i}}^{\theta_i} q_i^*(x, e)dx.$$

If we have $q_j(\theta, e) > 1 - e_j$ for a positive measure of types, then we must have $q_i(\theta, e) < 0$ for a positive measure of types by the aggregate feasibility constraint. Hence, if we reduced $q_j(\theta, e) = 1 - e_j$ for a positive measure of types, this wouldn't affect any constraints but instead increase profit as it allows us to increase $q_i(\theta, e) < 0$ for a positive measure of types, contradicting the optimality of the mechanism.

□

Now we can restate the problem as follows.

$$\max_{(q_i, t_i)_{i \in N}} \quad \sum_{i=1}^{n} \int_{\Theta} \Bigg[\int_{\{\theta_i | q_i(\theta_i, \theta_{-i}, e) \leq 0\}} q_i(x, \theta_{-i}, e)dx + \left(\theta_i q_i(\theta_i, \theta_{-i}, e) - \int_{\underline{\theta_i}}^{\theta_i} q_i(x, \theta_{-i}, e)dx \right) \Bigg] f(\theta)d\theta$$

s. t.

$q_i(\theta, e)$ is increasing in θ_i

$q_i(\theta, e) \geq -e_i$

$0 = \sum_{i=1}^{n} q_i(\theta, e)$

After some transformations[1], the problem above can be rewritten as follows:

[1] The details can be followed in Appendix D.

$$\max_{(q_i,t_i)_{i\in N}} \sum_{i=1}^{n} \left[\int_{\Theta} q_i(\theta_i,\theta_{-i},e)\left[\frac{\mathbb{1}\{q_i(\theta_i,\theta_{-i},e)\le 0\}}{f_i(\theta_i)} + \left(\theta_i - \frac{(1-F_i(\theta_i))}{f_i(\theta_i)}\right)\right] f(\theta)d\theta\right]$$

s. t.

$$q_i(\theta,e) \text{ is increasing in } \theta_i$$

$$q_i(\theta,e) \ge -e_i$$

$$0 = \textstyle\sum_{i=1}^{n} q_i(\theta,e_i)$$

Let $B_i(\theta_i) = \left(\theta_i - \frac{(1-F_i(\theta_i))}{f_i(\theta_i)}\right)$, which is the virtual value and $S_i(\theta_i) = \left(\theta_i + \frac{(F_i(\theta_i)}{f_i(\theta_i)}\right)$, which is the virtual cost. Virtual value and virtual cost are essentially the marginal revenue and marginal cost of having more buyers and sellers in the marketplace, respectively. We are going to focus on distributions that have increasing virtual values and costs.

Definition 2.1. The distribution of an agent i's type, F_i is **regular** if both B_i and S_i are increasing.

2.3 An Algorithm to Calculate the Optimal Allocation

I now describe an algorithm that would determine the allocation rule that maximizes the expected revenue.

Given some type profile, θ, suppose without loss of generality that $B_1(\theta_1) \ge \cdots \ge B_i(\theta_i) \ge \cdots \ge B_n(\theta_n)$ and that $S_{s(1)}(\theta_{s(1)}) \ge \cdots \ge S_{s(i)}(\theta_{s(i)}) \ge \cdots \ge S_{s(n)}(\theta_{s(n)})$ where $s(i)$ denotes the agent with i-th highest virtual cost.

Let us define auxiliary allocation functions among the agents: $q_i^k(\theta)$ denotes the allocation the agent i receives from agent k. We start the algorithm at $q_i^k(\theta) = 0$ for each $i, k \in N$.

We will define the allocation as follows.

Set $i = 1$ and $j = s(n)$.

1. If $B_i(\theta_i) \not> S_j(\theta_j)$, go to (3).

2. If $B_i(\theta_i) > S_j(\theta_j)$,

$$q_i^j(\theta) = \min\{1 - e_i - \sum_{\{k:s^{-1}(k)\le s^{-1}(j)\}} q_i^k(\theta), e_j + \sum_{k=1}^{i-1} q_j^k\}.$$

i. If

$$1 - e_i - \sum_{\{k : s^{-1}(k) \leq s^{-1}(j)\}} q_i^k(\theta) < e_j + \sum_{k=1}^{i-1} q_j^k$$

$i = i + 1$ and go to (1).

ii. If

$$1 - e_i - \sum_{\{k : s^{-1}(k) \leq s^{-1}(j)\}} q_i^k(\theta) > e_j + \sum_{k=1}^{i-1} q_j^k$$

$j = s(s^{-1}(j) - 1)$ and go to (1).

iii. If

$$1 - e_i - \sum_{\{k : s^{-1}(k) \leq s^{-1}(j)\}} q_i^k(\theta) = e_j + \sum_{k=1}^{i-1} q_j^k$$

$i = i + 1$ and $j = s(s^{-1}(j) - 1)$ and go to (1).

3. For each $k \in N$, let $q_k(\theta) = \sum_{a=1}^{n} q_k^a(\theta)$ and exit.

We are going to see that this algorithm does not admit any cycles.

Proposition 2.4. *The allocations algorithm described above does not admit any cycles.*

Proof. Given some type profile, θ, suppose without loss of generality that $B_1(\theta_1) \geq \cdots \geq B_i(\theta_i) \geq \cdots \geq B_n(\theta_n)$ and that $S_{s(1)}(\theta_{s(1)}) \geq \cdots \geq S_{s(i)}(\theta_{s(i)}) \geq \cdots \geq S_{s(n)}(\theta_{s(n)})$.

Suppose there was a cycle: $\mathcal{B}_i(\theta_i) \geq \mathcal{S}_j(\theta_j); \mathcal{B}_j(\theta_j) \geq \mathcal{S}_k(\theta_k), \ldots, \mathcal{B}_y(\theta_y) \geq \mathcal{S}_z(\theta_z)$ but $\mathcal{B}_z(\theta_z) > \mathcal{S}_i(\theta_i)$.

Notice that for any agent α, we have

$$\mathcal{S}_\alpha(\theta_\alpha) = \mathcal{B}_\alpha(\theta_\alpha) + \frac{1}{f_\alpha(\theta_\alpha)} \geq \mathcal{B}_\alpha(\theta_\alpha).$$

Then, above cycle implies that

$$\mathcal{B}_i(\theta_i) \geq \mathcal{S}_j(\theta_j) \geq \mathcal{B}_j(\theta_j) \geq \mathcal{S}_k(\theta_k), \ldots, \mathcal{B}_y(\theta_y) \geq \mathcal{S}_z(\theta_z) \geq \mathcal{B}_z(\theta_z) > \mathcal{S}_i(\theta_i),$$

a contradiction as it comes to mean $\mathcal{B}_i(\theta_i) > \mathcal{S}_i(\theta_i)$.

□

Now, we can prove the optimality of this algorithm in maximizing the platform's profit.

Theorem 2.1. *Suppose each agent's type is drawn from a regular distribution. Then, the revenue-maximizing mechanism has the allocation rule described by the above algorithm and the following transfer rule:*

$$t_i^*(\theta, e) = \int_{\{\theta_i | q_i(\theta_i, \theta_{-i}, e) \leq 0\}} q_i(x, \theta_{-i}, e)dx + \theta_i q_i(\theta, e) - \int_{\underline{\theta_i}}^{\theta_i} q_i(x, \theta_{-i}, e)dx.$$

Proof. Consider the point-wise maximization problem given (θ, e) and ignore the constraint that q_i is increasing in θ_i for a moment.

Claim 2.1. *Let i and j be such that $B_i(\theta_i) > S_j(\theta_j)$. If $q_j(\theta_j) > -e_j$, then $q_i(\theta, e) = 1 - e_i$.*

Proof. Let i and j be such that $B_i(\theta_i) > S_j(\theta_j)$. If $q_i(\theta, e) < 1 - e_i$ and $q_j(\theta, e) > -e_j$, then increasing $q_i(\theta, e)$ and decreasing $q_j(\theta, e)$ by the same amount, to the extent that it is possible under the constraints, strictly increases the revenue. □

Since the profit is the expected difference of virtual values and costs of trades that realize, the most profitable trade (per quantity) is the one between the agent with the highest virtual value and the one with the lowest virtual cost. If their trade has a negative virtual surplus, that means in that type profile, there is no profitable trade, so it is better to shut down. If their trade has a positive virtual surplus, then it is profitable for the marketplace to intermediate their trade, at the maximum capacity. Once their trade is accomplished, the new most profitable trade is again between the agent with the highest virtual value and the agent with the lowest virtual cost, among the remaining agents. Thus, it is optimal to proceed in this manner, until there is no more trade with positive virtual surplus.

□

3 Example with $n = 2$

Suppose there are two agents, $N = \{1, 2\}$ with endowments $(e_1, e_2) = (0.6, 0.5)$. Each θ_i is distributed uniformly over $[0, 1]$ with c.d.f. $F_i(\theta_i) = \theta_i$. Then, the virtual values and costs are given by

$$\mathcal{B}_i(\theta_i) = 2\theta_i - 1 \text{ and } \mathcal{S}_i(\theta_i) = 2\theta_i.$$

The Figure 2 depicts the space of (θ_1, θ_2) where the shaded areas represent $\mathcal{B}_1(\theta_1) > \mathcal{S}_2(\theta_2)$ and $\mathcal{B}_2(\theta_2) > \mathcal{S}_1(\theta_1)$ respectively. As you can see, a large part of the type spaces sees no trade. This is similar to a monopolist who finds it optimal to exclude some buyers from trade, to get more from the other ones. However, in this case, the agents can actually create a surplus by themselves as long as they have different valuations for the good (which happens with probability 1), since they each have some endowment and demand for the good. I study the problem of the marketplace that takes into account the possibility that agents can trade outside the marketplace in Idem (2021).

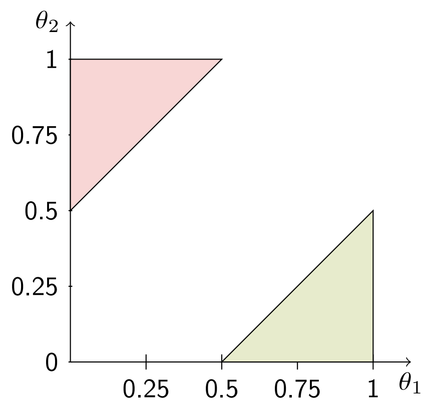

Figure 2: x-axis represents Θ_1 and y-axis represents Θ_2. Green and red areas show the type profiles at which agent 1 and 2 is the buyer, respectively.

We can also compare this to the welfare maximization of Myerson and Satterthwaite (1983). There, the trade would be one sided only and the mechanism would allow the trade as long as the buyer's value is greater than the seller's value plus 0.25. Suppose one of the agents have 1 unit of endowment and the other has none:

The trapezoid area between the large and the small triangle is, in a sense, the cost of profit rather than welfare maximization.

Straightforward calculations show that

$$\mathcal{B}_i(\theta_i) > \mathcal{S}_j(\theta_j) \iff \theta_i > \theta_j + 0.5.$$

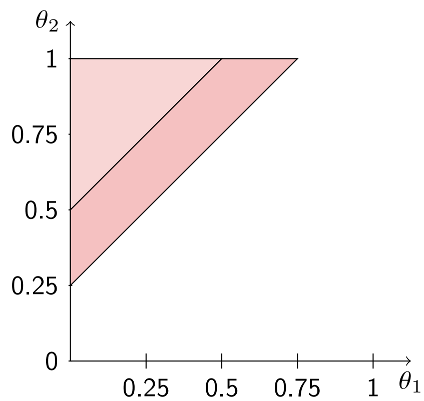

Figure 3: x-axis represents Θ_1 and y-axis represents Θ_2. The small triangle is the area where our mechanism would allow trade while the larger one is the area where Myerson and Satterthwaite (1983) would.

By Theorem 2.1, we have

$$q(\theta, e) = \begin{cases} (0.4, -0.4), & \text{if } \theta_1 > \theta_2 + 0.5, \\ (-0.5, 0.5), & \text{if } \theta_2 - 0.5 > \theta_1, \\ (0, 0), & \text{otherwise.} \end{cases}$$

Now, we can calculate the transfers.

$$t_1(\theta, e) = \begin{cases} 0.4(\theta_2 + 0.5), & \text{if } \theta_1 > \theta_2 + 0.5, \\ -0.5(\theta_2 - 0.5), & \text{if } \theta_2 - 0.5 > \theta_1, \\ 0, & \text{otherwise.} \end{cases}$$

$$t_2(\theta, e) = \begin{cases} -0.4(\theta_1 - 0.5), & \text{if } \theta_1 > \theta_2 + 0.5, \\ 0.5(\theta_1 + 0.5), & \text{if } \theta_2 - 0.5 > \theta_1, \\ 0, & \text{otherwise.} \end{cases}$$

4 Optimality Among Bayesian Mechanisms

Now, I will show that the restriction to the dominant-strategy incentive-compatible mechanisms is without loss. First, assuming the good is indivisible, I show that this is the case through a direct application of Gershkov et al. (2013). Then, I will assume that the endowments are rational numbers and show that the equivalence is true when the good is divisible as well. The restriction of the endowments to rational numbers is a reflection of realistic constraints: Every trade is limited by a finite decimal points for both quantities and the payments in virtually every venue. Thus, there is no reason to think an agent might come to the marketplace with the endowment $\frac{1}{\pi}$.

4.1 Indivisible Goods

Suppose the good is indivisible and can only be traded at increments of $d < 1$. Then, of course, each endowment and allocation must be an integer multiple of d. Clearly, the set of admissible allocations that allocate only integer multiples of d is finite. We can define each of these admissible allocations as separate outcomes: An outcome is a vector of allocations, $(o_1, \ldots, o_n)$ such that o_i is an integer multiple of k and $1 - e_i \geq o_i \geq -e_i$. Let O be the set of all such outcomes. (Notice that not all elements of O are feasible, as we didn't consider aggregate feasibility when constructing it.)

We can map these outcomes of Gershkov et al. (2013). There, they assume that there are finitely many outcomes, and agents' utilities under the outcome k are $u_i^k(x_i, t_i) = a_i^k x_i + c_i^k + t_i$ where $a_i^k, c_i^k \in \mathbb{R}$ are constants, x_i is an agent's private type and t_i is a monetary transfer. Then, for each outcome $o \in O$, we can define a corresponding outcome k in their environment such that for each i, $a_i^k = o_i$ and $c_i^k = 0$. Thus, their result about the equivalence of Bayesian and Dominant-Strategy Implementations apply to our environment, when the good is indivisible.

4.2 Divisible Goods

Now, suppose the endowments are rational numbers. We will first introduce an artificial divisibility constraint. Under this constraint, the result from Gershkov et al. (2013) will apply directly, with the same construction as above. Then, we will take the fractions that can be traded to the limit 0 and obtain the result for the divisible good.

Let $d^i = 2^{-i}$ for each $i = 1, 2, \ldots$. For each i, we define the outcome sets, O^i for each d^i, similar to above. These sets will again be finite for each $i < \infty$. Thus, as the construction above shows, Gershkov et al. (2013) implies that the Bayesian and Dominant-Strategy Implementations are payoff and profit equivalent. Let o^i be the allocation that is implemented when the trades can only be integer multiples of d^i. Moreover, let O^i be the Bayesian-optimal allocation under the same scenario. Then, the equivalence implies $O_j^i = \mathbb{E}_{-j}[o^i]$, for each agent j and for each $i = 1, 2 \ldots$. As we consider the limit $i \to \infty$, both sequences $(o_i)_{i=1}^{\infty}$ and $(O_i)_{i=1}^{\infty}$ converge. To see this, notice that since the endowments are feasible, the optimal allocations when the good is divisible will be rational numbers as well. Thus, they will have finite binary representations. Thus, after some finite integers $k^o, k^O \in \mathbb{N}$, the solution of the indivisible problem will become feasible for each case. Since that is the solution to the unconstrained problem, when it is feasible, it will be the solution to the constrained problem as well. Therefore, both sequences converge. Let o and O be their limits. Then, the continuity of the expectation implies that $O_j = \mathbb{E}_{-j}[o]$. Of course, this means that the solution to the dominant-strategy implementation problem with indivisible good is also a solution to the Bayesian implementation problem with indivisible good.

5 Marketplace as a Seller

A very natural extension of the model above is to allow the marketplace to be a seller as well. For instance, Amazon facilitates trade between buyers and sellers. However, it also sells some products itself, some of which are even produced by Amazon. Thus, here I allow the marketplace to be able to own some units at a per-unit cost, c. Let agent 0 be the marketplace with $q_0(\theta)$ denoting the units the marketplace itself sells at the type profile θ. Finally, let e_0 be the supply of the marketplace. Unlike other agents' endowments, I allow the marketplace to own more than 1 unit. Then, the problem is as follows:

$$\max_{(q_i,t_i)_{i\in N}} \sum_{i=1}^{n} \left[\int_{\Theta} q_i(\theta_i,\theta_{-i},e)\left[\frac{\mathbb{1}\{q_i(\theta_i,\theta_{-i},e)\le 0\}}{f_i(\theta_i)} + \left(\theta_i - \frac{(1-F_i(\theta_i))}{f_i(\theta_i)}\right)\right] f(\theta)d\theta\right] - c\int_{\Theta} cq_0(\theta)f(\theta)d\theta$$

s. t.

$$q_i(\theta, e) \text{ is increasing in } \theta_i$$

$$q_i(\theta, e) \ge -e_i$$

$$0 = \textstyle\sum_{i=1}^{n} q_i(\theta, e_i)$$

A simple modification of the algorithm will be enough to capture the new element. Let $\mathcal{S}_0 = c$. Then, the marketplace will enter itself to the ranking of the virtual costs with this constant virtual cost and everything else will work the same as before. Proof that this is the optimal allocation is omitted as it is essentially the same as before, except there is now one more potential seller. (Of course, if we interpret c as valuation rather than the cost of producing or obtaining the good, the marketplace can also end up as a buyer. Then, we would also define the virtual value as $\mathcal{B}_0 = c$.)

Notice that if another agent's valuation is equal to c as well, then this agent's virtual cost is higher than the virtual cost of the marketplace, since the virtual cost of the marketplace is equal to the actual cost but the agent's virtual cost include an additional positive term, $\frac{F(c)}{f(c)}$. This introduces some inefficiency: The marketplace prefers to sell its own endowment before an agent whose valuation is equal to c, since buying from an agent comes with the information rent. As I argued in the introduction, this is an area of active debate in the context of Amazon and other big tech companies. US Congress may take actions to regulate Amazon's use of third-party sales data to curate their own products (e.g. Amazon Basics) as well as their own first-party retail sales (e.g. books, electronics produced by other companies, etc.) (Congress Majority Staff, 2020).

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A Proof of Lemma 1

Proof. Let $\theta \in \Theta$ and $\theta_i' \in \Theta_i$. Then, by incentive compatibility

$$\theta_i \min\{1, q_i(\theta, e) + e_i\} - t_i(\theta_i, \theta_{-i}, e) \geq \theta_i \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} - t_i(\theta_i', \theta_{-i}, e)$$

and

$$\theta_i' \min\{1, q_i(\theta_i, \theta_{-i}, e) + e_i\} - t_i(\theta_i, \theta_{-i}, e) \leq \theta_i' \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} - t_i(\theta_i', \theta_{-i}, e).$$

Subtracting the second inequality from the first one leads to:

$$(\theta_i - \theta_i') \min\{1, q_i(\theta_i, \theta_{-i}, e) + e_i\} \geq (\theta_i - \theta_i') \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\}$$

Suppose $q(\theta, e) + e_i < 1$ and $\theta_i > \theta_i'$. Then, we have

$$\begin{aligned} \min\{1, q_i(\theta, e) + e_i\} &\geq \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} \iff \\ 1 > q_i(\theta, e) + e_i &\geq \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} \iff \\ q_i(\theta, e) &\geq q_i(\theta_i', \theta_{-i}, e) \end{aligned}$$

Now suppose $q(\theta, e) + e_i \geq 1$ and $\theta_i' \geq \theta_i$. Then,

$$\begin{aligned} \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} &\geq \min\{1, q_i(\theta, e) + e_i\} \iff \\ \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} &\geq 1 \iff \\ q_i(\theta_i', \theta_{-i}, e) + e_i &\geq 1. \end{aligned}$$

□

B Proof of Lemma 2

Proof. First, suppose $q_i(\theta, e) + e_i < 1$. Then, IC implies that for a type θ_i agent:

$$\begin{aligned} u_i(\theta, e) &= \max_{\theta_i' \in \Theta_i} \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\}\theta_i - t_i(\theta_i', \theta_{-i}, e) - e_i\theta_i \\ &= \max_{\theta_i' \in \Theta_i} q_i(\theta_i', \theta_{-i}, e)\theta_i - t_i(\theta_i', \theta_{-i}, e). \end{aligned}$$

Notice that the RHS is the maximum of affine functions of θ_i, so $u_i(\theta_i, e)$ is convex in θ_i in this region. Hence, $u_i(\theta_i, \theta_{-i}, e)$ is differentiable almost everywhere in θ_i on this region. For any θ_i at which it is differentiable, for $\delta > 0$, IC implies that

$$\begin{aligned} &\lim_{\delta\to 0} \frac{u_i(\theta_i + \delta, \theta_{-i}, e) - u_i(\theta, e)}{\delta} \\ \geq\ &\lim_{\delta\to 0} \frac{(q_i(\theta, e)(\theta_i + \delta) - t_i(\theta, e)) - (q_i(\theta, e)\theta_i - t_i(\theta, e))}{\delta} = q_i(\theta, e). \\ &\lim_{\delta\to 0} \frac{u_i(\theta, e) - u_i(\theta_i - \delta, \theta_{-i}, e)}{\delta} \\ \leq\ &\lim_{\delta\to 0} \frac{(q_i(\theta, e)\theta_i - t_i(\theta, e)) - (q_i(\theta, e)(\theta_i - \delta) - t_i(\theta, e))}{\delta} = q_i(\theta, e). \end{aligned}$$

Then, two inequalities together imply that

$$\frac{\partial u_i(\theta, e)}{\partial \theta_i} = q_i(\theta, e).$$

Now suppose $q_i(\theta, e) + e_i \geq 1$. Then,

$$u_i(\theta, e) = \min\{1, q_i(\theta, e) + e_i\}\theta_i - t_i(\theta, e) - e_i\theta_i = \theta_i - t_i(\theta, e) - e_i\theta_i.$$

Notice that $t_i(\theta, e)$ must be constant in θ_i on the region with $q_i(\theta, e) + e_i \geq 1$: Since agent i's effective allocation is constant, otherwise, i would simply choose the type with the least cost. Then, of course,

$$\frac{\partial u_i(\theta, e)}{\partial \theta_i} = 1 - e_i$$

□

C Proof of Proposition 1

Proof. We want to show that for each $i \in N$, for each $\theta_i, \theta_i' \in \Theta_i$ and $\theta_{-i} \in \Theta_i$, we have

$$\begin{aligned}
& u_i(\theta, e) \geq \theta_i \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} - t_i(\theta_i', \theta_{-i}, e) - \theta_i e_i \\
\iff \quad & u_i(\theta, e) \geq \theta_i \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} + \theta_i' \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} \\
& -\theta_i' \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} - t_i(\theta_i', \theta_{-i}, e) - \theta_i e_i + \theta_i' e_i - \theta_i' e_i \\
\iff \quad & u_i(\theta, e) \geq \theta_i \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} - \theta_i' \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} \\
& +u_i(\theta_i', \theta_{-i}, e) - e_i(\theta_i - \theta_i') \\
\iff \quad & u_i(\theta_i, \theta_{-i}, e) - u_i(\theta_i', \theta_{-i}, e) \geq (\theta_i - \theta_i') \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} \\
& -e_i(\theta_i - \theta_i') \\
\iff \quad & \int_{\theta_i'}^{\theta_i} q_i^*(x, \theta_{-i}, e) dx \geq \int_{\theta_i'}^{\theta_i} q_i^*(\theta_i', \theta_{-i}, e) dx
\end{aligned}$$

Suppose $\theta_i > \theta_i'$. Since $q_i^*(\cdot, \theta_{-i}, e)$ is increasing, $q_i^*(x, \theta_{-i}, e) \geq q_i^*(\theta_i', \theta_{-i}, e)$ for each $x \in [\theta_i', \theta_i]$. Then, the last inequality above holds. Similar analysis holds for the case of $\theta_i < \theta_i'$.

□

D Transformations of the Designer's Problem

We start with the following problem in Equation 2.2 and make the following transformation:

$$
\begin{aligned}
& \int_{\Theta} \int_{\underline{\theta_i}}^{\theta_i} q_i(x, \theta_{-i}, e) dx f(\theta) d\theta \\
= \quad & \int_{\Theta_{-i}} \int_{\Theta_i} \int_{x}^{\overline{\theta_i}} q_i(x, \theta_{-i}, e) f(\theta) d\theta dx \\
= \quad & \int_{\Theta_{-i}} \int_{\Theta_i} q_i(x, \theta_{-i}, e) \int_{x}^{\overline{\theta_i}} f_i(\theta_i) d\theta_i f_{-i}(\theta_{-i}) d\theta_{-i} dx \\
= \quad & \int_{\Theta} q_i(\theta_i, , \theta_{-i}, e) \left(\frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right) f(\theta) d\theta
\end{aligned}
$$

So, the second part of the objective function becomes:

$$
\begin{aligned}
\textstyle\sum_{i=1}^{n} \quad & \int_{\Theta} \theta_i q_i(\theta_i, \theta_{-i}, e) f(\theta) d\theta - \int_{\Theta} q_i(\theta_i, \theta_{-i}, e) \left(\frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right) f(\theta) d\theta \\
= \quad & \textstyle\sum_{i=1}^{n} \int_{\Theta} \left(\theta_i q_i(\theta_i, \theta_{-i}, e) - q_i(\theta_i, \theta_{-i}, e) \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right) f(\theta) d\theta \\
= \quad & \textstyle\sum_{i=1}^{n} \int_{\Theta} q_i(\theta_i, \theta_{-i}, e) \left(\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right) f(\theta) d\theta
\end{aligned}
$$

Next we look at the first summand in the objective function above. Notice that inside is actually a constant given θ_{-i} so it can be expressed as below:

$$
\begin{aligned}
& \int\limits_{\Theta} \left[\int\limits_{\{y | q_i(y, \theta_{-i}, e) \leq 0\}} q_i(x, \theta_{-i}, e) dx \right] f(\theta) d\theta \\
= \quad & \int\limits_{\Theta_{-i}} \left[\int\limits_{\{y | q_i(y, \theta_{-i}, e) \leq 0\}} q_i(x, \theta_{-i}, e) dx \right] f_{-i}(\theta_{-i}) d\theta_{-i} \\
= \quad & \int\limits_{\Theta_{-i}} \left[\int\limits_{\Theta_i} q_i(x, \theta_{-i}, e) \mathbb{1}\{q_i(x, \theta_{-i}, e) \leq 0\} dx \right] f_{-i}(\theta_{-i}) d\theta_{-i} \\
= \quad & \int\limits_{\Theta} q_i(\theta_i, \theta_{-i}, e) \frac{\mathbb{1}\{q_i(\theta_i, \theta_{-i}, e) \leq 0\}}{f_i(\theta_i)} f(\theta) d\theta
\end{aligned}
$$

Finally, the objective function can be written as:

$$
\begin{aligned}
& \textstyle\sum_{i=1}^{n} \left[\int\limits_{\Theta} q_i(\theta_i, \theta_{-i}, e) \frac{\mathbb{1}\{q_i(\theta_i, \theta_{-i}, e) \leq 0\}}{f_i(\theta_i)} f(\theta) d\theta \right] \\
& + \textstyle\sum_{i=1}^{n} \left[\int\limits_{\Theta} q_i(\theta_i, \theta_{-i}, e) \left(\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right) f(\theta) d\theta \right] \\
& = \textstyle\sum_{i=1}^{n} \left[\int_{\Theta} q_i(\theta_i, \theta_{-i}, e) \left[\frac{\mathbb{1}\{q_i(\theta_i, \theta_{-i}, e) \leq 0\}}{f_i(\theta_i)} + \left(\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right) \right] f(\theta) d\theta \right]
\end{aligned}
$$

Hence, the revenue maximization problem can be expressed as

$$\max_{(q_i,t_i)_{i\in N}} \sum_{i=1}^{n} \left[\int_{\Theta} q_i(\theta_i, \theta_{-i}, e) \left[\frac{\mathbb{1}\{q_i(\theta_i, \theta_{-i}, e) \le 0\}}{f_i(\theta_i)} + \left(\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right) \right] f(\theta) d\theta \right]$$

s. t.

$q_i(\theta, e)$ is increasing in θ_i

$q_i(\theta, e) \ge -e_i$

$0 = \sum_{i=1}^{n} q_i(\theta, e_i)$