# Existence of Competitive Equilibria in Convex Economies

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**Abstract**

In this paper, I show that the existence of a solution to a market design problem can be obtained as long as the designer's and the agents' preferences satisfy any sufficiently well-behaved abstract convexity, using 'convex' price orders that rank bundles instead of price vectors. Walrasian Equilibrium is obtained as a special case.

## 1 Introduction

Convexity has been one of the most common assumptions in many different lines of economic research, including showing the existence of equilibrium as well as in studying the efficiency, stability and other properties of the equilibria (Mas-Colell et al. (1995); Kreps (2012)). However, Euclidean convexity that we commonly refer to as convexity is not the only notion of betweenness.

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Even though the mathematical theory of abstract convexities has been developing for decades, economics has not benefited from these developments until recently. Although there were earlier attempts to incorporate these structures within economic frameworks (most notably Koshevoy (1999) and Nehring and Puppe (2007)), Richter and Rubinstein (2015) and Richter and Rubinstein (2017) introduced them to general equilibrium and decision theory contexts.

In the first part, I show that the existence of Nash equilibrium for generalized games can be extended to much more general convexities with appropriate modifications of the assumptions. Building on this result, I show the existence of competitive equilibrium under certain assumptions related to the interactions between the abstract convexity and the underlying topological space.

One important distinction to make is that in the equilibrium definition, price vector is replaced by a public ordering, as defined by Richter and Rubinstein (2015). The public ordering that clears the market orders the consumption bundles in the economy in a way that each individual maximizes her preferences when her budget is all the bundles rankled lower than her endowment. This generalizes the standard prices since in the standard economy, costs of consumption bundles define a public ordering such that the lower contour set of the endowment in this order is the budget set for each agent.

# 2 Abstract Convexities

## 2.1 Definitions and Familiar Results

We first introduce the concept of abstract convexity, then review and verify that many of the standard results about the standard convex preferences hold for abstract convex preferences as well with very similar proofs.[1] First, we want to define what is an abstract convexity and what it means to for a preference to be convex in this context. Preferences are complete and transitive binary relations in what follows unless otherwise stated.

We first give the general definition of convexity.

**Definition 2.1.** A family $\mathcal{C}$ of subsets of a set $X$ is called a **convexity** on $X$ if

(C-1) The empty set $\varnothing$ and the universal set $X$ are in $\mathcal{C}$;

(C-2) $\mathcal{C}$ is stable for intersections;

(C-3) $\mathcal{C}$ is stable for nested unions.

The pair $(X, \mathcal{C})$ is called a **convex structure**.

Notice that all three of above axioms of convexity are satisfied by the standard convex sets. Next, we define the convex hull for a convexity in the same way we do for the standard convexity.

**Definition 2.2.** Given a convex structure $(X, \mathcal{C})$, for $A \subset X$, **convex hull** of $A$ is defined as:

$$K(A) = \bigcap \{C | A \subset C \in \mathcal{C}\}. \tag{1}$$

[1] Most of the material about convexities discussed in this section can be found in van de Vel (1993b).

Moreover, a set $A$ is said to be convex if $A \in \mathcal{C}$ or equivalently $A = K(A)$.

Next, we define convex preferences more generally as was introduced by Richter and Rubinstein (2017).

**Definition 2.3.** A preference $\succeq$ is **convex** w.r.t. $\mathcal{C}$ if for any alternative $a \in X$, the strict upper contour set $U(\succeq, a) = \{x | x \succ a\}$ is convex w.r.t. $\mathcal{C}$.

Some simple but useful results has not been proven for these general convexities. So we prove them here to be prepared for the next section. The next definition introduces a measure of richness of the convexity by categorizing the convexities based on their polytopes. This will be useful in the next two lemmas.

**Definition 2.4.** A convex structure is of **arity** $\leqslant n$ provided its convex sets are precisely the sets $C$ with the property that $co\{F\} \subset C$ for each subset $F$ with $\#F \leqslant n$. (We will say of arity $n$ (rather than of arity at most $n$) for ease of reading unless the distinction is important.)

**Lemma 2.1.** Suppose the convexity on $X$ is of arity $n$. If $\succeq$ is convex, then the set of $\succeq$-best points in a convex set is convex.

*Proof.* Let $A \subset X$ and $B(\succeq, A)$ be the set of $\succeq$-best points in $A$. Assume that $A$ is convex. Suppose there are $n$ points $x_1, \ldots, x_n \in B(\succeq, A)$. Then, $x_i \sim x_j$ for each $i, j = 1, \ldots, n$. Let $z \in K(\{x_1, \ldots, x_n\})$. Since $\succeq$ is convex, the (weak) upper contour set is a convex set: $UC(\succeq, x) = K(UC(\succeq, x))$ and by monotonicity of the convex hull operator, $z \in UC(\succeq, x)$ so that $z \succeq x$. Also, since $x \in B(\succeq, A)$, for each $t \in A$, we have $x \succeq t$. Then, by transitivity, combining $z \succeq x$ and $x \succeq t$ yields $z \succeq t$ for each $t \in A$. Hence, $z \in B(\succeq, A)$ and $B(\succeq, A)$ is convex. □

**Definition 2.5.** Suppose the convexity on $X$ is of arity $n$. Let $f : X \to \mathbb{R}$. $f$ is **quasi-concave** if, $f(z) \geqslant \min\{f(x_1), \ldots, f(x_n)\}$ for each $z$ in $K(\{x_1, \ldots, x_n\})$.

**Lemma 2.2.** Suppose the convexity on $X$ is of arity $n$. Let $\succeq$ be preference on $X$ and a function $f : X \to \mathbb{R}$ represents it. $\succeq$ is convex, if and only if, $f$ is quasi-concave.

*Proof.* If $\succeq$ is convex, then the weak upper contour sets are convex: $UC(y, \succeq) = K(UC(y, \succeq))$ for each $y \in X$. Now, let $x_2, \ldots, x_n \in UC(y, \succeq)$. By monotonicity of the convex hull operator, $K(y, x_2, \ldots, x_n) \subset UC(y, \succeq)$. Then, for each $z \in K(y, x_2, \ldots, x_n)$, $z \succeq y$. So, $u(z) \geqslant u(y) = \min\{u(y), u(x_2), \ldots, u(x_n)\}$ so that $u(\cdot)$ is quasi-concave.

Conversely, if $u(\cdot)$ is quasi-concave, take any $y, x_2, \ldots, x_n \in X$ such that $f(x_2), \ldots, f(x_n) \geqslant f(y)$. Then, by quasi-concavity, $f(z) \geqslant f(y)$ for each $z \in K(y, x_2, \ldots, x_n)$ and hence upper contour set of $y$ is convex. □

### 2.1.1 A Specific Class of Convexities of Economic Interest

Richter and Rubinstein (2015, 2017) suggested a way to generate a convexity based on some *primitive orderings* in economic contexts:

**Definition 2.6.** Let $X$ be a set of outcomes and $\Lambda$ a set of (complete and transitive) binary relations that we refer to as **primitive orderings**. Then, convex hull operator of $\Lambda$-convexity on $X$ is defined as follows: For each $A \subset X$

$$K(A) = \{x | \forall \geqslant_k \in \Lambda, \exists a_k \in A \text{ s.t. } x \geqslant_k a_k\}. \tag{2}$$

**Remark 2.1.** Richter and Rubinstein (2015) establishes that for every convex preference with respected to a $\Lambda$-convexity, the weak upper contour sets are also convex: $WU(\succeq, a) = \{x | x \succeq a\}$.

Then, this gives rise to the following equivalent definition of a convex preference for a convexity generated by primitive orderings, characterized in Proposition 6 of Richter and Rubinstein (2017).

**Definition 2.7.** A preference $\succeq$ is **$\Lambda$-convex** if for any two alternatives $a, b \in X$; if for each $\geqslant_k \in \Lambda$, there is an alternative $y_k \in X$ with $y_k \neq b$ such that (i) $b \geqslant_k y_k$ and (ii) $y_k \succeq a$, then $b \succeq a$.

One motivation they provided is as follows: Suppose some professors are evaluating job market candidates based on research, teaching and charm. To convince them that candidate $b$ should be chosen rather than $a$, you would find for each criterion an alternative $c$ which is inferior to $b$ according to that criteria but they still prefer $c$ to $a$.

This approach encompasses the standard convexity as well. Let $X = \mathbb{R}^n$. Then, the orderings induced by all linear functionals in $\mathbb{R}^n$ would define the standard convexity. If we only consider the orderings induced by the strictly positive linear functionals, then this corresponds to a convexity such that if a preference is convex with respect to it, then it is strictly increasing.

## 2.2 Existence of a Nash Equilibrium

We are going to prove the existence of a competitive equilibrium by constructing a generalized game where a player's actions affect the other player's feasible actions. In this section, we prove the existence of a Nash equilibrium in a generalized game where the players have convex preferences and convex strategy spaces.

We present a generalization of Kakutani's fixed point theorem that replaces the standard convexity with an abstract one. Before stating the theorem, we give definitions of several concepts needed for this theorem:

- A Topological Convex Structure (tcs) is a set $X$ with a convexity $\mathcal{C}$ and a topology $\mathcal{T}$ such that all polytopes of $\mathcal{C}$ are closed in $\mathcal{T}$.
- In a convexity, a *half-space* is a convex set whose complement is also convex.

- A tcs $X$ is $S_4$ if for each pair of disjoint and non-empty convex sets $C, D$, there exists a half-space $H \subset X$ with $C \subset H$ and $D \subset X \backslash H$.
- A tcs $X$ is *closure stable* if closure of each set is also convex.
- a tcs $X$ is *properly locally convex* if each $x \in X$, has a *neighborhood base* of convex open sets. (I.e., a family of open and convex sets $\{N_\alpha\}$ such that every neighborhood $N$ of $x$ includes some member $N_\alpha$ of this family: $N_\alpha \subset N$.)
- A tcs $X$ is $FS_4$ if for each pair of disjoint and non-empty convex closed sets $C, D$, there exists a continuous CP functional of $X$ separating $C$ and $D$.

**Lemma 2.3** (Theorem 6.15 of Chapter 4.6, van de Vel (1993b))**.** Let $X$ be a compact Hausdorff tcs with connected convex sets. Let $F : X \Rightarrow X$ be a nonempty-, convex-, closed-valued upper hemi-continuous correspondence. If either $X$ is properly locally convex, closure stable and $S_4$ or $X$ is $FS_4$, then $F$ has a fixed point.

**Remark 2.2.** To be able to use Berge's theorem, we need a metric space anyway and metric spaces are Hausdorff topological spaces. In what follows, we restrict ourselves to compact, properly locally convex, closure stable and $S_4$ metric spaces with connected convex sets. We could have considered $FS_4$ spaces as well but we do not take that route and focus on generalizations of Euclidean spaces here for now.

We will need the following lemma when we construct the "joint best responses" correspondence and apply Lemma 2.3.

**Lemma 2.4.** Consider some tcs $\{X_1, \ldots, X_n\}$. If each $X_i$ is a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets, the their product tcs $X = \Pi_{i=1}^n$ is also a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets.

*Proof.* Consider the product tcs of some tcs $\{X_1, \ldots, X_n\}$ which are compact, properly locally convex, closure stable and $S_4$ metric spaces with connected convex sets. It is a standard result from topology that compactness and connectedness is preserved under products. Closure stability follows from theorem 1.10 of chapter 3.1 in van de Vel (1993b). The product is $S_4$ by theorem 3.15 of chapter 1.3 in van de Vel (1993b). Finally, it is properly locally convex which can be seen by taking the product of neighborhood base in each dimension. □

We record the version of Berge's Theorem (or Theorem of Maximum) we will be using next.

**Lemma 2.5.** Let $X$ and $Y$ be metric spaces and $f : X \times Y \Rightarrow \mathbb{R}$ be a continuous function. Suppose that $F : X \Rightarrow Y$ is a correspondence that is upper and lower hemi-continuous, and compact- and nonempty-valued. Then, the optimal choice correspondence $Z : X \Rightarrow Y$ defined by $Z(x) = \arg\max_{y \in F(x)} f(x, y)$ is upper hemi-continuous and the maximum value function $m : X \Rightarrow \mathbb{R}$ is continuous.

Finally, we give formal definitions of a generalized game and a Nash equilibrium for such a game.

**Definition 2.8.** An n-player generalized game $G = (\{A_i, C_i, u_i\}_{i=1}^n)$ consists of, for each $i = 1, \ldots, n$,

(i) A set of actions $A_i$;

(ii) A constraint (feasibility) correspondence $C_i : A_{-i} \Rightarrow A_i$ (where $A_{-i} = \Pi_{j \in \{1,\ldots,n\} \setminus \{i\}} A_j$);

(iii) A utility function $u_i : \Pi_{j=1}^{n} A_j \Rightarrow \mathbb{R}$.

**Definition 2.9.** A Nash equilibrium for a generalized game $G = (\{A_i, C_i, u_i\}_{i=1}^{n})$ is a strategy profile $(a_i^*)_{i=1}^{n} \in \Pi_{i=1}^{n} A_i$ such that, for each $i = 1, \ldots, n$,

(i) $a_i^* \in C_i(a_{-i}^*)$;

(ii) $a_i^*$ maximizes $u(a_i, a*_{-i})$ over $a_i \in C_i(a_{-i}^*)$.

Now, we can prove the existence of Nash equilibrium in a generalized game.

**Theorem 2.1.** Suppose that $G = (\{A_i, C_i, u_i\}_{i=1}^{n})$ is a generalized game for which

(i) Each $A_i$ be a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets;

(ii) Each $C_i$ is a continuous, non-empty-, closed- and convex-valued correspondence;

(iii) Each $u_i$ is jointly continuous in $a \in A = \Pi_{j=1}^{n} A_j$ and represents a convex preference $\succeq_i$.

Then, $G$ has a Nash equilibrium.

*Proof.* Consider the problem of maximizing the real-valued $u_i(a_i, a_{-i})$ with respect to $a_i \in C_i(a_{-i})$ for each $a_{-i} \in A_{-i}$ and for each $i \in N$. Then, applying Berge's theorem directly thanks to assumptions made above, we see that the best response correspondences $A_i^* : A_{-i} \Rightarrow A_i$ are non-empty- and compact-valued, and upper hemi-continuous.

Take any $x \in \mathbb{R}$. Since we assumed that $u_i$ represents a convex preference, its argmax on a convex set is convex by lemma 2.1. Hence, $A^*$ is convex-valued for each player.

Define the joint best response correspondence $A^* : A \Rightarrow A$ as follows: At each $a \in A$, we have $b \in A^*(a)$ if for each $i \in N$, $b_i \in A^*(b_{-i})$. It inherits the properties of individual best responses by lemma 2.4 and hence it is a non-empty-, convex-, closed-valued and upper hemi-continuous correspondence in a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets. Hence, by Kakutani's theorem 2.3, $G$ has a fixed point and by construction, the fixed point is a Nash equilibrium. □

**Corollary 2.1.** Suppose that $G = (\{A_i, C_i, u_i\}_{i=1}^n)$ is a generalized game for which

(i) Each $A_i$ be a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets;

(ii) Best Response of each individual, $A_i^*(a_{-i})$ is an upper hemi-continuous, non-empty-, closed- and convex-valued correspondence.

Then, $G$ has a Nash equilibrium.

*Proof.* The proof is same as the last step of the previous proposition: Define the joint best response correspondence $A^* : A \Rightarrow A$ as follows: At each $a \in A$, we have $b \in A^*(a)$ if for each $i \in N$, $b_i \in A^*(b_{-i})$. It inherits the properties of individual best responses by lemma 2.4 and hence it is a non-empty-, convex-, closed-valued and upper hemi-continuous correspondence in a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets. Hence, by Kakutani's theorem 2.3, $G$ has a fixed point and by construction, the fixed point is a Nash equilibrium. □

## 2.3 Topology and Convexity of Space of Public Orderings

As it is common in the general equilibrium theory (see for example Kreps (2012)), our existence result will employ an auctioneer who chooses a 'price' to clear the markets. However, unlike the standard framework where the set of strategies of the auctioneer can be reduced to the unit simplex of relevant dimension, it is a complicated object in this setting: The auctioneer chooses a public ordering. Notice that this is a generalization of the standard economy with prices. For example, with linear prices, possible price vectors are non-negative linear functionals, which are among primitive orderings of the standard economy. Moreover, linear prices also define an ordering on the set of allocations in the natural way: If $p \cdot x \geqslant p \cdot y$ then $x \succeq_p y$.

Showing that the set of strategies of the auctioneer is well-behaved enough to be used in Theorem 2.1 requires quite a lot of work and involves defining a convexity and topology on public orderings that are consistent with the individual's consumption spaces' convexities and topologies, then verifying that this convexity and topology inherits topological and convexity related properties of each individual's consumption space. Therefore, we reserve this for the appendix and summarize the results here.

**Assumption 2.1.** We assume that for an economy $\mathcal{E}$, the consumption sets of individuals are same: $X_i = X_j$ for each $i, j = 1, \ldots, n$.

This assumption ensures that we have a clear interpretation of the public ordering. If we don't make this assumption, the public ordering would be comparing the consumption bundles in spaces distinct from each other. Then, there would be situations in which an agent can afford a bundle according to the public ordering but this bundle is not even in his consumption space. We can deal with this mathematically but it would make little sense to do it that way.

An alternative, which is mathematically feasible is to define the public ordering as the union of public orderings of each individual's consumption space. I.e., $P \subset \cup_{i=1}^{n} X_i \times X_i$. Everything we are doing here can be done with this with appropriate adjustments but it would still be more difficult to interpret such an equilibrium with such a public ordering.

It is worth noting that we do not assume the consumption spaces of individuals to be endowed with the same convexity or topology. Even though the sets $X_i$ are the same, we keep using the subscript to distinguish them from their product and to emphasize the individual convexities and topologies.

The following lemma which has been proven in the appendix shows that the set of public orderings of an auctioneer satisfies the assumptions of 2.1.

**Lemma 2.6.** Let $X_i$ be a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets. Then, the space of closed subsets of $X_i^2$ can be endowed with the Hausdorff metric, Vietoris topology and Vietoris convexity. Moreover, this tcs is also a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets.

## 2.4 Feasible Trades and Admissible Public Orderings

We haven't yet specified the set of feasible trades. We do it implicitly by introducing a set of feasible allocations $\mathcal{F} \subset X = \Pi_{i=1}^{n} X_i$, given a profile of initial endowments. (Since the endowments will be fixed, we do not explicitly show the dependence of the set to the endowments in our notation.) We denote the set of admissible public orderings for an economy by $\mathcal{P}$ as defined in Appendix B.

We revise the definition of Walrasian equilibrium accordingly.

**Definition 2.10.** A Walrasian equilibrium for an economy $\mathcal{E} = \langle N, (X_i, \geq_i, e^i)_{i=1}^{n}, \mathcal{F} \rangle$ is a public ordering $p \in \mathcal{P}$ and an allocation profile $x \in X$ such that

(i) For each consumer $i$, $x^i$ solves the problem: maximize $u_i(y^i)$ subject to $y^i \in X_i$ and $e^i \geqslant_p y^i$;

(ii) Markets clear: $x \in \mathcal{F}$.

**Assumption 2.2.** Given an economy $\mathcal{E} = \langle N, (X_i, \succeq_i, e^i)_{i=1}^n, \mathcal{F} \rangle$, there exists a preference $\succeq_A$ on $X \times \mathcal{P}$ such that

(i) It is represented by a continuous function $u_A(x, p)$ that is quasi-concave in $p$;

(ii) If $x \in X$ with $x \notin \mathcal{F}$ and $q \in \arg\max_{p \in \mathcal{P}} u(x, p)$, then there exists $i \in N$ such that $x^i >_q e^i$.

This assumption ensures that there is a well-behaved preference on the set of allocation and public ordering bundles such that, if some allocation is not feasible from some profile of initial endowments, then any public ordering that maximize this preference given this allocation has the property that there is at least one individual who cannot afford his allocation under this ordering.

Even though this higher order assumption looks obscure, considering the fact that the set of feasible allocation profiles is entirely arbitrary in this model, this is a necessary restriction. In the standard setting, the Walrasian auctioneer's preferences are continuous and quasi-concave in prices and the second part of the assumption is also satisfied thanks to linearity of the prices. Thus, this assumption is simply a generalization of what is implied by the linear price structure in the standard case.

In proving the existence of a Walrasian equilibrium under standard convexity, we consider an artificial auctioneer whose utility function is the value of the excess demand. In that case, it is of course possible for the auctioneer to find a price vector that would make some individual violate his budget, if the allocation profile is infeasible.[2] This assumption is an abstract counterpart to it. Thus, it serves a similar purpose and it will help us show the market clearance.

## 2.5 Existence of Competitive Equilibrium

The following lemma shows the continuity of the budget sets in public orderings and it has been proved in the appendix Appendix A.1.

**Lemma 2.7.** Consider an economy $\mathcal{E} = \langle N, (X_i, \succeq_i, e^i)_{i=1}^n, \mathcal{F} \rangle$ where $X_i$ is a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets and it is identical for each individual.[3] Let $p$ be a continuous public ordering. Then, $B_i(p) = \{x \in X_i | e^i \geqslant_p x\}$ is continuous.

Now, we give an abstract version of previous Walrasian equilibrium existence theorem.

**Proposition 2.1.** Consider an economy $\mathcal{E} = \langle N, (X_i, \succeq_i, e^i)_{i=1}^n, \mathcal{F} \rangle$. Assume:

(i) For each $i, j \in N$, $X_i = X_j$ and $X_i$ is a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets.

(ii) $\succeq_i$ is continuous and convex. (We take $u_i$ to represent $\succeq_i$ for each individual $i$ in $N$.)

(iii) There exists a utility function $u_A$ on $X \times \mathcal{P}$ such that it is continuous in both arguments and quasi-concave in $p \in \mathcal{P}$; and if $x \in X$ with $x \notin \mathcal{F}$ and if $q$ maximizes $\succeq_A$ given $x$, then there exists $i \in N$ such that $x^i >_q e^i$.

[2]Indeed, the last step of the standard proof of existence of competitive equilibrium would also show this, using its contrapositive Kreps (2012).

[3]Notice that we require the set of possible allocations $X_i$ to be identical for each agent but we do not require the associated topologies and convexities to be the same. Moreover, we do not rule out the possibility that some bundles are not available for some agents, it is implicitly embedded in the definition of feasible allocation profiles, $\mathcal{F}$.

Then, $\mathcal{E}$ has a Walrasian equilibrium.

*Proof.* Consider the following generalized game.

- The players are the individuals in $N$ and an auctioneer.
- The strategy space $\mathcal{P}$ of the auctioneer is a non-empty convex compact subset of $c(X_i^2)$ such that each $p \in \mathcal{P}$ is a continuous, concave[4] and reflexive ordering on $X_i$.
- Each individual $i$'s strategies are constrained by choice of the auctioneer: He must choose $x^i$ from the set $\{x \in X_i | e^i \geqslant_p x\}$ where $\geqslant_p$ is the public ordering that the auctioneer chose. When $i$ chooses $x$, his utility is $u_i(x)$.
- We endow the auctioneer with the preference $\succeq_A$.

Each individual's strategy space has been assumed to be a non-empty tcs such that it is a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets, so the first condition of the previous proposition holds for them.

The auctioneer's strategy space is also given similarly as shown by lemma 2.6.

The auctioneer's feasibility correspondence is constant (hence continuous) and is equal to a compact, convex, non-empty set everywhere and hence it satisfies the second condition of the previous proposition.

The continuity and convexity of preferences of individuals is assumed.

The continuity of preferences of the auctioneer is also assumed.

Now, the feasibility correspondences of the individuals (as well as the auctioneer) are clearly convex (by concavity of the public ordering), compact

[4]By a concave ordering, I mean one whose lower contour sets are convex.

(by continuity of the public ordering and the fact that $X_i$ is compact) and nonempty (by reflexivity of the public ordering) everywhere. The trouble is continuity and it has been shown in the Appendix A.1.

Then, Theorem 2.1 applies and there is a Nash equilibrium $(p, (x^i))$ of this generalized game. We want to show that this is a Walrasian equilibrium of this economy. Utility maximization is obvious by construction. We need to verify that markets clear.

Now, suppose $x \notin \mathcal{F}$. Then, we know that the best response of the auctioneer would require him to choose a public ordering such that at least one individual $i$ cannot afford $x^i$ given his endowment $e^i$ under this public ordering. So, there cannot be a Nash equilibrium in which $x \notin \mathcal{F}$. Hence, the Nash equilibrium must be a Walrasian equilibrium of this economy.

$\square$

# Appendix A Continuity of Budget Sets

Consider an economy $\mathcal{E} = \langle N, (X_i, \succeq_i, e^i)_{i=1}^n, \mathcal{F}\rangle$ where $X_i$ is a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets; and consumption spaces are identical for individuals.

We define the budget sets $B_i(p) = \{x \in X_i | e^i \geqslant_p x\}$. In this section, we want to show the continuity of these sets in the public ordering, $p$. We first give the relevant definitions and results that we are going to use.

A closed-valued correspondence (between metric spaces $A, B$) $F : A \Rightarrow B$ is *upper hemi-continuous*, if at each $a \in A$, for each $a_n \in A^\infty$ that converges to $a$ and for each $b_n \in B^\infty$ with $b_n \in F(a_n), \forall n = 1, \ldots$ that converges to $b$, $b \in F(a)$.

A correspondence (between metric spaces $A, B$) $F : A \Rightarrow B$ is *lower hemi-continuous*, if at each $a \in A$, for each $a_n \in A^\infty$ that converges to $a$ and for each $b \in F(a)$, there exists a subsequence $a_{n_k}$ and a sequence $b_k \in B^\infty$ that converges to $b$ with $b_k \in F(a_{n_k})$, for each $k$.

A correspondence is *continuous* if it is both upper hemi-continuous and lower hemi-continuous.

Now, we define the (Kuratowski) convergence of a sequence of sets. Let $Z$ be a topological space. Let $\{Z_n\}$ be a sequence of subsets of $Z$.

$\limsup Z_n$ is the set of elements $z$ such that there is a sequence of elements $z_k$ and a subsequence $\{Z_{n_k}\}$ such that $z_k \in Z_{n_k}$ and $z_k$ converges to $z$.

$\liminf Z_n$ is the set of elements $z$ such that there is a sequence of elements $z_n$ with $z_k \in Z_n$ and $z_n$ converges to $z$.

$Z' = \lim Z_n$ if $\liminf Z_n = Z_n \limsup Z_n$. If this is the case, $Z_n$ converges to $Z'$ in Kuratowski sense.

Consider $(c(X_i^2), T_V)$. This is a compact Hausdorff space, since it is a compact metric space, with the Hausdorff metric. Then, by Theorem 4.7 in Illanes and Nadler (1999), convergence in Kuratowski sense and Convergence in $T_V$ are equivalent on $c(X_i^2)$. Now, we can prove the following lemma.

**Lemma Appendix A.1.** Consider an economy $\mathcal{E} = \langle N, (X_i, \succeq_i, e^i)_{i=1}^n, \mathcal{F} \rangle$ where $X_i$ is a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets and identical for each individual. Let $p$ be a continuous public ordering. Then, $B_i(p) = \{x \in X_i | e^i \geqslant_p x\}$ is continuous.

*Proof.* Upper hemi-continuity: Suppose $\{p_m\}$ is a sequence of public orderings in $P$ approaching $p$ and $x^i_m$ is feasible at $p_m$ for each $m$ and $\lim_m x^i_m = x^i$. By feasibility, $x^i_m \in X_i$ for each $m$, so by compactness, $x^i \in X_i$. Again, by feasibility, $e^i \geqslant_{p_m} x^i_m$. Now, it is known that in compact metric spaces, Kuratowski convergence and convergence in Hausdorff metric coincide. Then, since $\{p_m\}$ converges to $p$, (the inner limit) $\liminf p_m$ is equal to $p$ as a set, since $p$ is the limit and hence is equal to inner and outer limit by definition. Now, since $e^i \geqslant_{p_m} x^i_m$, we have $(x^i_m, e^i) \in p$ with $(x^i, e^i)$ being its limit. Then, $(x^i, e^i)$ is in the $\liminf p_m$ and hence in $p$. Hence, $e^i \geqslant_p x^i$, yielding upper hemi-continuity.

Lower hemi-continuity: Let $p_m$ be a sequence of public orderings that converges to some public ordering $p$ and let $x^i$ be a feasible allocation for individual $i$ with the endowment $e^i$ under $p$. (So, $e^i \geqslant_p x^i$ or equivalently, $(x^i, e^i) \in p$.) We want to show that there is a subsequence $p_{m_k}$ of $p_m$ and a sequence of allocations $x^i_k$ such that it converges to $x^i$ and for each $k$, $e^i \geqslant_{p_k} x^i_k$. Now, using the fact that $(x^i, e^i) \in p$ where $p$ is the limit of $p_m$ and hence is equal to the outer limit, by definition of the outer limit, we have a sequence of points $(x^i_k)$ and a subsequence of sets $p_{m_k}$ of $p_m$ such that $(x^i_k, e) \in P_{m_k}$ with $(x^i_k, e^i)$ converging to $(x^i, e^i)$. The subsequence of public orderings $p_{m_k}$ and the sequence of allocations $x^i_k$ with the stated qualities was precisely what we needed for lower hemi-continuity. $\square$

# Appendix B Space of Public Orderings

Assume throughout this section that $X_i$ is a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets.

Notation: $X = \Pi_{i=1}^n X_i$ and $X_i^2 = X_i \times X_i$. We call $x_i \in X_i$ an *allocation* and $x \in X$ an *allocation profile*.

Given the metrics on the individual's consumption spaces, $X_i$, we endow $X_i^2$ with the *Manhattan metric*: Let $d_i$ be the metric on $X_i$. Then, we define:

$$d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i), \forall x, y \in X_i^2. \tag{3}$$

Let $c(X_i^2)$ be the set of (nonempty) compact subsets of $X_i^2$. We define the *Hausdorff metric* on $c(X_i^2)$ as follows:

$$d_H(A, B) = \max\{\max_{z \in A} \min_{t \in B}\{d(z, t)\}, \max_{z \in B} \min_{t \in A}\{d(z, t)\}\}, \forall A, B \in c(X_i^2). \tag{4}$$

It is routine to check that both of these metrics are well-defined and indeed, metrics on their respective spaces.

Let $(Y, T)$ be an arbitrary topological space and let $cl(Y)$ be the set of (nonempty) closed sets in $Y$. Then, the Vietoris topology on $cl(Y)$ is the smallest topology $T_V$ such that

(i) If $U \in T$, then $\{A \in CL(Y) | A \subset U\} \in T_V$.

(ii) If B is $T$-closed, then $\{A \in cl(Y) | A \subset B\}$ is $T_V$-closed.

Let the topology generated by $d_H$ be denoted by $T_H$. Also, notice that $cl(X_i^2) = c(X_i^2)$ since closed subsets of a compact space is compact. Now, by theorem 3.1 in Illanes and Nadler (1999), $T_V = T_H$. Moreover, by theorem 3.5 there, $(c(X_i^2), T_V)$ is compact.

Next, we define a convexity on $c(X_i^2)$. To do this, we first take the product convexity on $X_i$ as the convexity on $X_i^2$. We need a couple of definitions.

Let $\mathcal{C}$ be a collection of subset of a set $Z$. We say that $(Z, \mathcal{C})$ is a *closure structure*, if (i) $\mathcal{C}$ includes the empty set and $Z$, (ii) $\mathcal{C}$ is closed under (arbitrary) intersections.

We say that a collection $\mathcal{S}$ of subset of a set $Z$ is a *subbase* for a convex structure $(Z, \mathcal{C})$, if $\mathcal{S} \subset \mathcal{C}$ and $\mathcal{C}$ is the coarsest convexity that includes $\mathcal{S}$.

Let $Z$ be a set and $A_1, \ldots, A_n \subset Z$. Define $\langle A_1, \ldots, A_n \rangle = \{B \subset 2^Z | B \subset \cup_{i=1}^n A_i; \forall i = 1, \ldots, n, B \cap A_i \neq \varnothing\}$.

Let $(Z, \mathcal{D})$ be a closure structure; $\mathcal{D}_* \equiv \mathcal{D} \backslash \{\varnothing\}$. Then, the sets $\langle D \rangle \cap \mathcal{D}_*$ and $\langle D, Z \rangle \cap \mathcal{D}_*$ for $D \in \mathcal{D}$ generate the *Vietoris Convexity* on the set $\mathcal{D}_*$.

Let $\mathcal{H}_*$ be the set of all (nonempty) closed, convex subset of $X_i^2$. Then, by definition, it defines a Vietoris convexity on $X_i^2$, $\mathcal{C}_V$.

We can indeed do this as by Proposition 3.10.4 of Chapter 3.3 in van de Vel (1993b), Vietoris metric and Vietoris convexity are compatible.[5]

By 3.7 in van de Vel (1993a), this space is $S_4$ and has connected convex sets, inheriting the properties of $X_i^2$, which inherits properties of $X_i$. Theorem 2.6 in van de Vel (1993a) implies that this space is properly locally convex and Theorem 2.4 there implies that it is closure stable. Hence, the space of (nonempty) closed subsets of $X_i^2$ is a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets. We summarize these results in the following lemma for future reference.

**Lemma Appendix B.1.** Let $X_i$ be a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets. Then, the

[5]This follows since when the space is compact, the convex closure of union of two compact convex set is also compact, as argued by van de Vel (1984) (Remark 1.7).

space of closed subsets of $X_i^2$ can be endowed with the Hausdorff metric, Vietoris topology and Vietoris convexity. Moreover, this tcs is also a compact, properly locally convex, closure stable and $S_4$ metric space with connected convex sets.

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