Online Appendix for Coexistence of Centralized and Decentralized Markets

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A Prohibitive Search Frictions

In the case of prohibitive search frictions, agents cannot trade outside the marketplace. Thus, the outside option is receiving 0 net utility from trade for each agent. The profit of the marketplace is the expected net payments. Thus, the designer seeks to maximize total payments, given incentive compatibility, individual rationality, and feasibility constraints.

$$
\begin{array}{lll}
\max\limits_{(q,t)} & \mathbb{E}_\theta\left[t(\theta)\right] & \\
\text{s. t.} & & \\
\text{(IC)} & \theta\min\{1,q(\theta)\}-t(\theta) & \geq \theta\min\{1,q(\theta',\theta)\}-t(\theta') \\
\text{(IR)} & \theta\min\{1,q(\theta)\}-t(\theta) & \geq 0 \\
\text{(Individual Feasibility)} & q(\theta) & \geq -1 \\
\text{(Aggregate Feasability)} & \mathbb{E}_\theta\left[q(\theta)\right] & \leq 0
\end{array}
$$

In the Section A.2 below, I develop a series of lemmata to simplify this problem. They allow the problem to be restated in terms of the virtual values and virtual costs, defined as follows.

Definition A.1. An agent with reported valuation θ has virtual value, $\mathcal{V}(\theta)$, and virtual cost, $\mathcal{C}(\theta)$, given by:

$$\mathcal{V}(\theta)=\theta-\frac{(1-F(\theta))}{f(\theta)} \text{ and } \mathcal{C}(\theta)=\theta+\frac{(F(\theta)}{f(\theta)}.$$

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Virtual value and virtual cost can be thought of as the marginal revenue and marginal cost. When an agent is a seller, that is, when an agent has a negative allocation, $q(\theta) < 0$, his deduction from the profit of the marketplace is the virtual cost. Similarly, when an agent is a buyer, $q(\theta) > 0$, his contribution to the profit is the virtual value. Then, the problem can be restated in these terms as follows:

$$\begin{aligned}
\max_{q(\cdot)} \quad & \mathbb{E}\left[q(\theta)\left(\mathbb{1}\{q(\theta) < 0\}C(\theta) + \mathbb{1}\{q(\theta) > 0\}\mathcal{V}(\theta)\right)\right] \\
\text{s. t.} \quad & \\
& q(\theta) \text{ is increasing} \\
& q(\theta) \geq -1 \\
& \mathbb{E}_\theta\left[q(\theta)\right] = 0
\end{aligned}$$

Definition A.2. The distribution of agents' valuations, F is **regular** if both $\mathcal{V}$ and C are increasing.

The regularity condition guarantees that the marketplace has a decreasing marginal revenue from having additional buyers and increasing marginal cost from having additional sellers. With this definition, we are ready to state the main result of this section, which characterizes the baseline optimal marketplace.

Proposition A.1. *Suppose the distribution F is regular. Then, the optimal mechanism has the allocation rule*

$$q(\theta) = \begin{cases} -1 & \text{if } \theta \leq \underline{\theta} \\ 0 & \text{if } \underline{\theta} < \theta < \overline{\theta} \\ 1 & \text{if } \theta \geq \overline{\theta} \end{cases}$$

and the transfer rule

$$t(\theta) = \begin{cases} -\underline{\theta} & \text{if } \theta \leq \underline{\theta} \\ 0 & \text{if } \underline{\theta} < \theta < \overline{\theta} \\ \overline{\theta} & \text{if } \theta \geq \overline{\theta} \end{cases}$$

where $\underline{\theta}$ and $\overline{\theta}$ satisfies $C(\underline{\theta}) = \mathcal{V}(\overline{\theta})$ and solves the problem

$$\begin{aligned} \max_{\underline{\theta},\overline{\theta}} \quad & \left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta}))\right] \\ s.\ t. \quad & \\ & F(\underline{\theta}) = 1 - F(\overline{\theta}) \\ & 0 \leq \underline{\theta} \leq \overline{\theta} \leq 1. \end{aligned}$$

Notice that each agent below $\underline{\theta}$ sells 1 unit and gets paid $\underline{\theta}$, and each agent above $\overline{\theta}$ buys 1 unit and pays $\overline{\theta}$. If the designer posts $\underline{\theta}$ as the price for selling and $\overline{\theta}$ as the price for buying, and let agents choose what to do, the allocation above represents exactly what the agents would do. Thus, the designer can implement this mechanism in a very straightforward way by posting bid-ask prices.

Proof of Proposition A.1. Since the allocation needs to be increasing, if $q(\theta) < 0$ for some θ, we would have $q(\theta') < 0$ for each $\theta' \leq \theta$. Similarly, if $q(\theta) > 0$ for some θ, we would have $q(\theta') > 0$ for each $\theta' \geq \theta$. So, let $0 \leq \underline{\theta} \leq \overline{\theta} \leq 1$ such that $\underline{\theta}$ is the supremum of values with negative allocation and $\overline{\theta}$ is the infimum of the values with positive allocation. Then, we can write the objective function as follows:

$$\begin{aligned} \Pi^M &= \mathbb{P}[\theta \in [0, \underline{\theta}]]\mathbb{E}[C(\theta)q(\theta)|\theta \in [0, \underline{\theta}]] + \mathbb{P}[\theta \in [\overline{\theta}, 1]]\mathbb{E}[\mathcal{V}(\theta)q(\theta)|\theta \in [\overline{\theta}, 1]] \\ &= \int_0^{\underline{\theta}} C(x)q(x)f(x)dx + \int_{\overline{\theta}}^1 \mathcal{V}(x)q(x)f(x)dx \end{aligned}$$

Note that for $\theta \leq \underline{\theta}$, we must have $q(\theta) = -1$ as -1 is the only possible negative allocation with the indivisible good and similarly, $q(\theta) = 1$ for $\theta \geq \overline{\theta}$. Thus, the optimal allocation will have the following form and the next step is to choose the cutoffs, $\underline{\theta}$ and $\overline{\theta}$ optimally.

$$q(\theta) = \begin{cases} -1 & \text{if } \theta \leq \underline{\theta} \\ 0 & \text{if } \underline{\theta} \leq \theta \leq \overline{\theta} \\ 1 & \text{if } \theta \geq \overline{\theta} \end{cases}$$

Then, the problem can be restated as follows:

$$\max_{\underline{\theta},\overline{\theta}} \quad \left[-\int_0^{\underline{\theta}} \mathcal{C}(x) f(x) dx + \int_{\overline{\theta}}^1 \mathcal{V}(x) f(x) dx \right]$$

s. t.

$$F(\underline{\theta}) = 1 - F(\overline{\theta})$$
$$0 \leq \underline{\theta} \leq \overline{\theta} \leq 1$$

Integrating out the total virtual value and the virtual cost using integration by parts shows that the objective function is equal to:

$$-\underline{\theta} F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta})) = F(\underline{\theta})(\overline{\theta} - \underline{\theta})$$

where the equality is obtained by using the feasibility condition $F(\underline{\theta}) = 1 - F(\overline{\theta})$. Then, there exists a solution to this problem. Moreover, the solution is interior: If $\underline{\theta} = 0$ or $\underline{\theta} = \overline{\theta}$, the profit is 0. However, positive profit is feasible by any feasible interior solution as is clear from the objective function.

Moreover, by using the formula for the transfer rule and the optimal allocation from above, we can compute the transfers in the mechanism to be

$$t(\theta) = \begin{cases} -\underline{\theta} & \text{if } \theta \leq \underline{\theta} \\ 0 & \text{if } \underline{\theta} \leq \theta \leq \overline{\theta} \\ \overline{\theta} & \text{if } \theta \geq \overline{\theta} \end{cases}$$

Notice that each 'seller' gets the same payment while each 'buyer' pays the same amount. Thus, this mechanism is equivalent to offering bid-ask prices that the transfer rule above suggest, that is a price for buying and a price for selling, and letting agents choose whether they want to buy or sell or not trade.

Finally, notice that the solution must have $\mathcal{C}(\underline{\theta}) = \mathcal{V}(\overline{\theta})$: If we had $\mathcal{C}(\underline{\theta}) > \mathcal{V}(\overline{\theta})$, decreasing $\underline{\theta}$ and adjusting $\overline{\theta}$ accordingly for feasibility would increase the profit since buying from $\underline{\theta}$ is costlier than what selling to $\overline{\theta}$ pays off. Similarly, if we had $\mathcal{C}(\underline{\theta}) < \mathcal{V}(\overline{\theta})$, then increasing $\underline{\theta}$ and adjusting $\overline{\theta}$ so that the feasibility binds would again increase the profit since there are more agents whose trade is profitable. □

A.1 Illustrative Example with Uniform Distribution

Suppose θ is distributed uniformly over $[0,1]$ with c.d.f. $F(\theta) = \theta$. Then, the virtual values and costs are given by

$$\mathcal{V}(\theta) = 2\theta - 1 \text{ and } C(\theta) = 2\theta\,.$$

It is easy to show that the objective function becomes $\underline{\theta}(1 - 2\underline{\theta})$ after substituting for $\overline{\theta} = 1 - \underline{\theta}$ (feasibility). So, the optimal cutoffs are $\underline{\theta} = \frac{1}{4}$ and $\overline{\theta} = \frac{3}{4}$.

Let us consider two agents with valuations θ_1 and θ_2. Then, 1 will buy from 2 in the optimal marketplace when it operates on its own if and only if $\theta_1 \geq 0.75$ and $\theta_2 \leq 0.75$, and 2 will buy from 1 if and only if $\theta_2 \geq 0.75$ and $\theta_1 \leq 0.75$. The Figure 1 depicts the space of (θ_1, θ_2) where the shaded areas represent these trading regions.

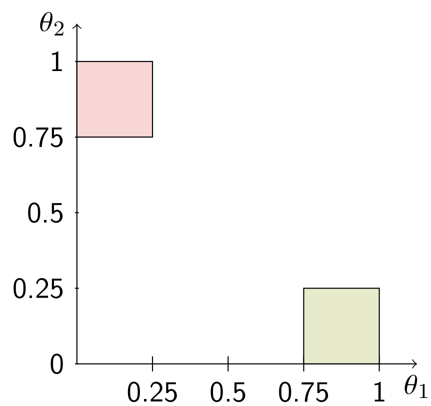

Figure 1: x-axis represents $\theta_1 \in [0,1]$ and y-axis represents $\theta_2 \in [0,1]$. Green and red areas show the type profiles at which agent 1 and 2 is the buyer, respectively.

Using the payment formula from the theorem above, we can compute the payments as below:

$$t(\theta) = \begin{cases} -\dfrac{1}{4}, & \text{if } \theta \leq \dfrac{1}{4}\,, \\ 0, & \text{if } \dfrac{1}{4} \leq \theta \leq \dfrac{3}{4}\,, \\ \dfrac{1}{4}, & \text{if } \theta \geq \dfrac{3}{4}\,. \end{cases}$$

The large area of the type space where there is no trade motivates our consideration for the decentralized market. The marketplace excludes these type profiles from trade because their is not profitable; this is akin to a monopolist or a monopsonist excluding some agents from trade. Indeed, the marketplace acts as both a monopolist and a monopsonist. However, unlike the buyers a monopolist excludes, the agents the marketplace excludes actually have a surplus that they can create, if they were allowed to trade. Thus, assumption that they will remain in the marketplace even though they are not trading is not very realistic. This is why in the general model, I allow them to choose between the marketplace and a decentralized market.

A.2 Simplifying the Problem

The results below have first been obtained in Idem (2021) for an environment with finitely many agents, divisible goods and arbitrary endowments. Here I restate them for the environment I study in the Coexistence of Centralized and Decentralized Markets, with the proofs adjusted accordingly. Moreover, here I focus on the case where the decentralized trade is not possible due to prohibitive frictions. This simplifies the individual rationality constraints to the usual form where the mechanism needs to guarantee nonnegative utility. The main text deals with the complications arising from endogenous outside option created by the search market participants.

I describe the setup and the initial statement of the mechanism design problem here for convenience.

- Good: There is a single, indivisible good in the market.
- Agents: There is a continuum of agents on $[0, 1]$.
- Endowments: Each agent has 1 unit of endowment of the good.
- Demands: Each agent demands up to 2 units of the good. Since the good is indivisible, this means, they can consume 0, 1, or 2 units, depending on whether they buy or sell, or neither buy nor sell.
- Valuations: Each agent has some valuation $\theta \in [0, 1]$ for a unit of the good. The valuations are drawn from some distribution F with support $[0, 1]$. Agents' valuations are their private information.
- Marketplace: A mechanism designer wants to design a mechanism to maximize its profit. She knows the distribution of valuations, F.

By revelation principle, I focus on direct mechanisms. Moreover, as agents are symmetric other than their valuations, I focus on anonymous mechanisms, which is without loss. Then, the designer will choose a mechanism that allocates $q : \theta \to \mathbb{R}$ units of good to each agent with valuation θ and asks her to pay $t : \theta \to \mathbb{R}$. Hence, the net utility of the agent with the valuation θ from the monagorastic mechanism is

$$u(\theta) = \theta \min\{1, q(\theta)\} - t(\theta).$$

As agents have demands for two units, having more than 2 unit of the good is same as having 2 unit. As such, the expression for the utility above caps the maximum net trade that increases the utility at 1, since the agent already has 1 unit of endowment.

The profit of the marketplace is the net payments. Thus, the designer seeks to maximize total payment, given the incentive compatibility, individual rationality, and feasibility constraints.

$$
\begin{array}{lll}
\max\limits_{(q,t)} \displaystyle\int_{[0,1]} t(\theta) f(\theta) d\theta & & \\
\text{s. t.} & & \\
\text{(IC)} & \theta \min\{1, q(\theta)\} - t(\theta) & \geq \theta \min\{1, q(\theta')\} - t(\theta') \\
\text{(IR)} & \theta \min\{1, q(\theta)\} - t(\theta) & \geq 0 \\
\text{(Individual Feasibility)} & q(\theta) & \geq -1 \\
\text{(Aggregate Feasability)} & \displaystyle\int_{[0,1]} q(\theta) f(\theta) d\theta & \leq 0
\end{array}
$$

We first develop a series of lemmata that help us state the maximization problem above as a concave program.

Lemma A.1 (Monotonicity). *Suppose (q, t) is a direct, IC mechanism. Then,*

1. *If $q(\theta) < 1$ for some $\theta \in [0, 1]$, then $q(\theta)$ is increasing at (θ).*
2. *If $q(\theta) \geq 1$ for some $\theta \in [0, 1]$, then $q(\theta') \geq 1$ for each $\theta' \geq \theta$.*

The proof is standard, except for taking care of the capacities.

Proof. Let $\theta, \theta' \in [0, 1]$. Then, by incentive compatibility

$$\theta \min\{1, q(\theta)\} - t(\theta) \geq \theta \min\{1, q(\theta')\} - t(\theta')$$

and

$$\theta' \min\{1, q(\theta)\} - t(\theta) \leq \theta' \min\{1, q(\theta')\} - t(\theta').$$

Subtracting the second inequality from the first one leads to:

$$(\theta - \theta') \min\{1, q(\theta)\} \geq (\theta - \theta') \min\{1, q(\theta')\}$$

Suppose $q(\theta) < 1$ and $\theta > \theta'$. Then, we have

$$
\begin{array}{lll}
\min\{1, q(\theta)\} & \geq \min\{1, q(\theta')\} & \Longleftrightarrow \\
1 > q(\theta) & \geq \min\{1, q(\theta')\} & \Longleftrightarrow \\
q(\theta) & \geq q(\theta') &
\end{array}
$$

Now suppose $q(\theta) \geq 1$ and $\theta' \geq \theta$. Then,

$$\begin{aligned}
\min\{1, q(\theta')\} &\geq \min\{1, q(\theta)\} \iff \\
\min\{1, q(\theta')\} &\geq 1 \iff \\
q(\theta') &\geq 1.
\end{aligned}$$

□

The next lemma presents the derivative of the utility of an agent in an IC mechanism.

Lemma A.2 (Envelope Condition). *If (q, t) is a direct, IC mechanism, then for each $\theta \in [0, 1]$*

$$\frac{\partial u(\theta)}{\partial \theta} = \begin{cases} q(\theta), & \text{if } q(\theta) < 1, \\ 1, & \text{otherwise.} \end{cases}$$

Again, the proof is similar to standard arguments.

Proof. First, suppose $q(\theta) < 1$. Then, IC implies that for type θ agent:

$$\begin{aligned}
u(\theta) &= \max_{\theta' \in [0,1]} \min\{1, q(\theta')\}\theta - t(\theta') \\
&= \max_{\theta' \in [0,1]} q(\theta')\theta - t(\theta').
\end{aligned}$$

Notice that the RHS is the maximum of affine functions of θ, so $u(\theta)$ is convex in θ on this region. Hence, $u(\theta)$ is differentiable almost everywhere in θ on this region. For any θ at which it is differentiable, for $\delta > 0$, IC implies that

$$\begin{aligned}
&\lim_{\delta \to 0} \frac{u(\theta + \delta) - u(\theta)}{\delta} \\
\geq \; &\lim_{\delta \to 0} \frac{(q(\theta)(\theta + \delta) - t(\theta)) - (q(\theta)\theta - t(\theta))}{\delta} = q(\theta). \\
&\lim_{\delta \to 0} \frac{u(\theta) - u(\theta - \delta)}{\delta} \\
\leq \; &\lim_{\delta \to 0} \frac{(q(\theta)\theta - t(\theta)) - (q(\theta)(\theta - \delta) - t(\theta))}{\delta} = q(\theta).
\end{aligned}$$

Then, two inequalities together imply that

$$\frac{\partial u(\theta)}{\partial \theta} = q(\theta).$$

Now suppose $q(\theta) \geq 1$. Then,

$$u(\theta) = \min\{1, q(\theta)\}\theta - t(\theta) = \theta - t(\theta).$$

Notice that $t(\theta)$ must be constant in θ on the region with $q(\theta) \geq 1$: Since agent's effective allocation is constant, otherwise, i would simply choose the type with the least cost. Then, of course,

$$\frac{\partial u(\theta)}{\partial \theta} = 1.$$

□

Notation: For any direct mechanism (q, t), let

$$q^*(\theta) = \begin{cases} q(\theta), & \text{if } q(\theta) < 1, \\ 1, & \text{otherwise.} \end{cases}$$

Note that for a direct, IC mechanism, $q^*(\theta)$ is also weakly increasing.

The next lemma gives the representation of the utility of each type as the integral of the allocation rule, using the previous lemma.

Lemma A.3 (Payoff Equivalence)**.** *If (q, t) is a direct, IC mechanism, then*

$$u(\theta) = u(0) + \int_0^\theta q^*(x)dx,$$

for each $\theta \in [0.1]$.

Proof. Since $u(\theta)$ is convex θ on both regions where $q(\theta) > 1$ and $q(\theta) \leq 1$ separately, it is absolutely continuous in θ. Then, it is the integral of its derivative. □

Next, we pin down the transfer rule in an IC mechanism.

Lemma A.4 (Revenue Equivalence)**.** *If (q, t) is a direct, IC mechanism, then*

$$t(\theta) \quad = -u(0) + \theta q^*(\theta) - \int_0^\theta q^*(x)dx,$$

for each $\theta \in [0, 1]$.

Proof. From the definition of $u(\theta)$ and the previous lemma. □

Now we show that the necessary conditions above for incentive compatibility of a mechanism are also sufficient to establish the incentive compatibility of a mechanism.

Lemma A.5. *Let (q, t) be a direct mechanism. The mechanism is incentive compatible if and only if,* *1.* $q^*(\theta)$ *is increasing at* θ*;*

2. $t(\theta) = -u(0) + \theta q^*(\theta) - \int\limits_0^\theta q^*(x)dx.$

Proof. We have already shown that an incentive compatible mechanism satisfies 1 and 2 above. To see the converse, suppose a mechanism satisfies 1 and 2. We want to show that for each $\theta, \theta' \in [0,1]$, we have

$$\begin{aligned}
& u(\theta) \geq \theta \min\{1, q(\theta')\} - t(\theta') \\
\iff \quad & u(\theta) \geq \theta \min\{1, q(\theta')\} + \theta' \min\{1, q(\theta')\} \\
& -\theta' \min\{1, q(\theta')\} - t(\theta') \\
\iff \quad & u(\theta) \geq \theta \min\{1, q(\theta')\} - \theta' \min\{1, q(\theta')\} \\
& +u(\theta') \\
\iff \quad & u(\theta) - u(\theta') \geq (\theta - \theta') \min\{1, q(\theta')\} \\
\iff \quad & \int_{\theta'}^{\theta} q^*(x)dx \geq \int_{\theta'}^{\theta} q^*(\theta')dx
\end{aligned}$$

Suppose $\theta > \theta'$. Since $q^*(\cdot)$ is increasing, $q^*(x) \geq q^*(\theta')$ for each $x \in [\theta', \theta]$. Then, the last inequality above holds. Similar analysis holds for the case of $\theta < \theta'$.

□

The next result shows that deterministic incentive compatible mechanisms are bid-ask price mechanisms.

Proposition A.2. *If* (q,t) *is a direct, deterministic and incentive compatible mechanism, then there are two types* $\underline{\theta}$ *and* $\overline{\theta}$ *and prices* p_s*,* p_0*, and* p_b *such that*

$$q(\theta) = \begin{cases} -1 & \textit{if } \theta \leq \underline{\theta} \\ 0 & \textit{if } \underline{\theta} < \theta < \overline{\theta} \\ 1 & \textit{if } \theta \geq \overline{\theta} \end{cases}$$

and the transfer rule

$$t(\theta) = \begin{cases} -p_s & \textit{if } \theta \leq \underline{\theta} \\ p_0 & \textit{if } \underline{\theta} < \theta < \overline{\theta} \\ p_b & \textit{if } \theta \geq \overline{\theta} \end{cases}$$

Proof. The form of the allocation follows from the fact that the good is indivisible, the mechanism is deterministic, and that the incentive compatible mechanisms have monotone allocations. Given this, integrating the revenue equivalence formula shows that for each allocation outcome, there is a unique price. □

Remark A.1. Notice that we haven't made any use of the individual rationality constraints; what we have so far are the implications of the incentive compatibility and the indivisibility of the object. Thus, even though we stated and proved the Proposition A.2 for the case of prohibitive search frictions, it applies to the case of coexistence as well.

The next proposition provides the characterization of the IR mechanisms by establishing the types with the lowest utilities. The reason this is an issue in this model is that in an auction, the lowest allocation an agent could receive is 0. Hence, the utility is always increasing in agent's type, as can be seen from the envelope condition. Of course, this means the lowest type has the lowest utility. However, here, an agent with a relatively low type can be a seller, which means he would get a negative allocation. Therefore, the utility of the lowest type is not the lowest utility, which can again by seen from the envelope condition.

Proposition A.3. *Let (q, t) be a direct IC mechanism. Then, it is IR if and only if,*

$$\theta^* q^*(\theta^*) \geq t(\theta^*),$$

where θ^* is defined as

1. *$\theta^* = 0$ if $q^*(0) \geq 0$,*
2. *$\theta^* = 1$ if $q^*(1) < 0$,*
3. *a solution to $q^*(\theta^*) = 0$ if such a type exists,*
4. *θ such that for each $\theta' < \theta$, $q(\theta') < 0$ and for each $\theta' > \theta$, $q(\theta') > 0$.*

Proof. Case 1: Suppose $q^*(0) \geq 0$. Then, by Lemma A.3, incentive compatibility of a mechanism implies that the associated ex-post utilities $u(\theta)$ are increasing in θ. Hence, if $u(0) \geq 0$, we have $u(\theta) \geq 0$ for each $\theta \in [0, 1]$.

Case 2: Suppose $q^*(1) < 0$. Then, by Lemma A.3, $u(\theta)$ are decreasing and hence, $u(1)$ is the lowest payoff. Hence, if it is nonnegative, all other types' payoffs are nonnegative as above.

Cases 3 and 4: Suppose θ^* is defined as in the Case 3 or Case 4. Then, by Lemma A.3, $u(\theta)$ is decreasing up to θ^* and increasing after that point. Hence, type θ^* has the lowest payoff. So, if $u(\theta^*) \geq 0$, each type's IR condition must also hold.

□

Lemma A.6. *If an IC and IR mechanism maximizes the expected revenue of the designer, then,*

$$t(\theta^*) = \theta^* q(\theta^*)$$

where θ^* is defined as

1. $\theta^* = 0$ *if* $q^*(0) \geq 0$,
2. $\theta^* = 1$ *if* $q^*(1) < 0$,
3. *the solution to* $q^*(\theta^*) = 0$ *if such a type exists,*
4. θ *such that for each* $\theta' < \theta$, $q(\theta') < 0$ *and for each* $\theta' > \theta$, $q(\theta') > 0$.

Proof. The previous proposition shows that IC and IR mechanisms must have $\theta^* q(\theta^*)$ greater than $t(\theta^*)$. However, if $\theta^* q(\theta^*) > t(\theta^*)$, then the seller can increase the expected revenue by increasing $t(0)$ and keeping the allocation rule the same. This would increase all types' payments and the revenue strictly, contradicting revenue maximization.

□

Using the condition about θ^* from Lemma A.6 and the previous lemmata, we have

$$\begin{aligned}
\theta^* q^*(\theta^*) &= t(\theta^*) \\
= &\quad -u(0) + \theta^* q^*(\theta^*) - \int_0^{\theta^*} q^*(x)dx \\
\iff &\quad u(0) = -\int_0^{\theta^*} q^*(x)dx \\
\iff &\quad t(\theta) = \int_0^{\theta^*} q^*(x)dx + \theta q^*(\theta) - \int_0^{\theta} q^*(x)dx
\end{aligned}$$

Now we are ready to show that the allocation rule in a revenue-maximizing mechanism is not 'wasteful'.

Proposition A.4. *Let (q, t) be a direct mechanism that maximizes the revenue of the designer. Then, $q(\theta) \leq 1$ with probability 1 and the aggregate feasibility holds with equality:* $\int_{[0,1]} q(\theta) f(\theta) d\theta = 0$.

Proof. First, suppose that in the optimal mechanism, there exists a set $\Theta \subset [0, 1]$ with a positive measure such that for each $\theta \in \Theta$, $q(\theta) > 1$. Notice that decreasing the allocation to 1 unit has no effect on the agent's payoff. Hence, it doesn't effect any IC or IR constraints.

Next, let us examine the transfer rule in a direct, IC mechanism:

$$t(\theta) = \int_0^{\theta^*} q^*(x)dx + \theta q^*(\theta) - \int_0^{\theta} q^*(x)dx.$$

If we have $q(\theta) > 1$ for a positive measure of types, then we must have $q(\theta) < 0$ for a corresponding positive measure of types by the aggregate feasibility constraint. Hence, if we reduced $q(\theta) = 1$ for $\theta \in \mathbf{\Theta}$, this wouldn't affect any constraints but instead strictly increase profit as it allows us to increase $q(\theta) < 0$ for a positive measure of types, contradicting the optimality of the mechanism.

By the same argument, having $\int\limits_{[0,1]} q(\theta)f(\theta)d\theta < 0$ cannot be optimal: Either buying less from types or selling more to some types would increase their payments, strictly increasing the profit. □

Now, by fixing $t(\theta)$ to the characterization we have from above, we can restate the problem as follows.

$$\max_q \quad \int\limits_{[0,1]} \left[\int\limits_{\{y|q(y)\leq 0\}} q(x)dx + \left(\theta q(\theta) - \int\limits_0^{\theta} q(x)dx \right) \right] f(\theta)d\theta$$

s. t.

$q(\theta)$ is increasing

$q(\theta) \geq -1$

$\int\limits_{[0,1]} q(\theta)f(\theta)d\theta = 0$

We make the following transformation:

$$\begin{aligned}
& \int\limits_{[0,1]} \int\limits_0^{\theta} q(x)dx f(\theta)d\theta \\
= & \int\limits_{[0,1]} \int\limits_x^{\overline{\theta}} f(\theta)d\theta q(x)dx \\
= & \int\limits_{[0,1]} q(x)(1-F(x))dx \\
= & \int\limits_{[0,1]} q(\theta) \left(\frac{(1-F(\theta))}{f(\theta)} \right) f(\theta)d\theta
\end{aligned}$$

So, the second part of the objective function becomes:

$$
\begin{aligned}
& \int_{[0,1]} \theta q(\theta) f(\theta) d\theta - \int_{[0,1]} q(\theta) \left(\frac{(1 - F(\theta))}{f(\theta)} \right) f(\theta) d\theta \\
= & \int_{[0,1]} \left(\theta q(\theta) - q(\theta) \frac{(1 - F(\theta))}{f(\theta)} \right) f(\theta) d\theta \\
= & \int_{[0,1]} q(\theta) \left(\theta - \frac{(1 - F(\theta))}{f(\theta)} \right) f(\theta) d\theta
\end{aligned}
$$

Next we look at the first summand in the objective function above. Notice that inside is actually a constant, so it can be expressed as below:

$$
\begin{aligned}
& \int_{[0,1]} \left[\int_{\{y | q(y) \leq 0\}} q(x) dx \right] f(\theta) d\theta \\
= & \int_{\{y | q(y) \leq 0\}} q(x) dx \\
= & \int_{[0,1]} q(x) \mathbb{1}\{q(x) \leq 0\} dx \\
= & \int_{[0,1]} q(\theta) \frac{\mathbb{1}\{q(\theta) \leq 0\}}{f(\theta)} f(\theta) d\theta
\end{aligned}
$$

Finally, the objective function can be written as:

$$
\begin{aligned}
& \int_{[0,1]} q(\theta) \frac{\mathbb{1}\{q(\theta) \leq 0\}}{f(\theta)} f(\theta) d\theta + \int_{[0,1]} q(\theta) \left(\theta - \frac{(1 - F(\theta))}{f(\theta)} \right) f(\theta) d\theta \\
= & \int_{[0,1]} q(\theta) \left[\frac{\mathbb{1}\{q(\theta) \leq 0\}}{f(\theta)} + \left(\theta - \frac{(1 - F(\theta))}{f(\theta)} \right) \right] f(\theta) d\theta
\end{aligned}
$$

Hence, the revenue maximization problem can be expressed as

$$
\begin{aligned}
& \max_{q} \quad \left[\int_{[0,1]} q(\theta) \left[\frac{\mathbb{1}\{q(\theta) \leq 0\}}{f(\theta)} + \left(\theta - \frac{(1 - F(\theta))}{f(\theta)} \right) \right] f(\theta) d\theta \right] \\
& \text{s. t.} \\
& \qquad q(\theta) \text{ is increasing} \\
& \qquad q(\theta) \geq -1 \\
& \qquad \int_{[0,1]} q(\theta) f(\theta) d\theta = 0
\end{aligned}
$$

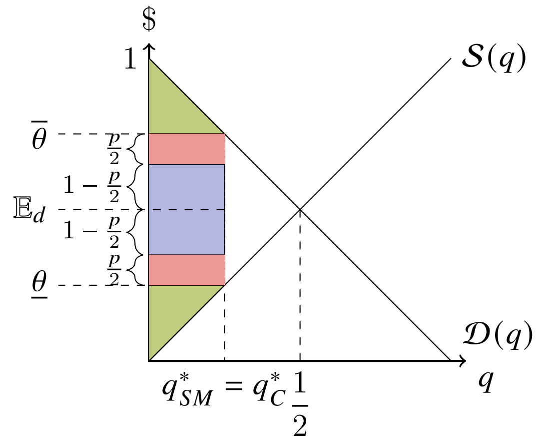

Figure 2: The ratio of the compensations only depend on p and not the distribution.

B Simple Economics of Optimal Marketplaces

Here is how Figure 2 works. The aggregate compensations for buyers is equal to $\frac{p}{2}q^* \times (\underline{\theta} - \mathbb{E}_d)$ where $\mathbb{E}_d$ the expected valuation in the decentralized market, i.e., $\mathbb{E}_d = \mathbb{E}[\theta | \theta \in [\underline{\theta}, \overline{\theta}]]$. Similarly, the aggregate compensations for sellers is equal to $\frac{p}{2}q^* \times (\mathbb{E}_d - \underline{\theta})$. Since $\mathbb{E}_d$ is a point between $\underline{\theta}$ and $\overline{\theta}$, this means total compensations will be equal to $\dfrac{p}{2}$ times the area of the rectangle between $\underline{\theta}$ and $\overline{\theta}$ on the vertical axis and between 0 and q^*_C; $(\overline{\theta} - \underline{\theta}) \times q^*_C$.

C Individual Rationality and the Allocations

Proof of Lemma 2. Notice that just below $\underline{\theta}$, the utility from the mechanism should have a left derivative below $-\frac{p}{2}$:

$$
\begin{aligned}
u^m(\theta) = u^m(0) + \int_0^{\theta} q(x)dx \geq u^d(\theta) \iff& \\
u^d(\underline{\theta}) - \int_0^{\underline{\theta}} q(x)dx + \int_0^{\theta} q(x)dx \geq u^d(\theta) \iff& \\
u^d(\underline{\theta}) - \int_{\theta}^{\underline{\theta}} q(x)dx \geq u^d(\theta) \iff& \\
p\left[\frac{\int\limits_{\underline{\theta}}^{\overline{\theta}} xf(x)dx}{2\left(F(\overline{\theta}) - F(\underline{\theta})\right)} - \frac{\underline{\theta}}{2}\right] - \int_{\theta}^{\underline{\theta}} q(x)dx \geq p\left[\frac{\int\limits_{\underline{\theta}}^{\overline{\theta}} xf(x)dx}{2\left(F(\overline{\theta}) - F(\underline{\theta})\right)} - \frac{\theta}{2}\right] \iff& \\
p\frac{\theta - \underline{\theta}}{2} \geq \int_{\theta}^{\underline{\theta}} q(x)dx \iff& \\
p\frac{\theta - \underline{\theta}}{2} \geq u^m(\underline{\theta}) - u^m(\theta).&
\end{aligned}
$$

Since only possible allocations are −1, 0, and 1, this means we must have $q(\underline{\theta}) = -1$. Moreover, due to monotonicity of the allocation, for each $\theta \in [0, \underline{\theta}]$, this implies $q(\theta) = -1$. Following the same steps around $\overline{\theta}$ also shows that $q(\theta) = 1$ for each $\theta \in [\overline{\theta}, 1]$. □

D Slope of Utilities from Search Market

Lemma D.1. *Under any equilibrium,* $-\frac{p}{2} \leq \frac{\partial u^d(\theta)}{\partial \theta} \leq \frac{p}{2}$ *for each agent.*

Proof. Suppose Θ^d is the set of agents who join the decentralized market, $\mu(\Theta^d) = \mathbb{P}[x \in \Theta^d]$ their measure, and let $\theta \in Cov(\Theta^d)$. Then,

$$
\begin{aligned}
u^d(\theta) &= \frac{p}{2}\mathbb{P}[x > \theta | x \in \Theta^d] \left[\mathbb{E}[x | x > \theta, x \in \Theta^d] - \theta\right] \\
&+ \frac{p}{2}\mathbb{P}[x < \theta | x \in \Theta^d] \left[\theta - \mathbb{E}[x | x < \theta, x \in \Theta^d]\right] \\
&= \frac{p}{2}\frac{\mathbb{P}[x > \theta, x \in \Theta^d]}{\mu(\Theta^d)} \left[\mathbb{E}[x | x > \theta, x \in \Theta^d] - \theta\right] \\
&+ \frac{p}{2}\frac{\mathbb{P}[x < \theta, x \in \Theta^d]}{\mu(\Theta^d)} \left[\theta - \mathbb{E}[x | x < \theta, x \in \Theta^d]\right] \\
&= \frac{p}{2\mu(\Theta^d)}\left[\int\limits_{\{x \in \Theta^d : x > \theta\}} x f(x) dx - \theta\mathbb{P}[x > \theta, x \in \Theta^d]\right] \\
&+ \frac{p}{2\mu(\Theta^d)}\left[\theta\mathbb{P}[x < \theta, x \in \Theta^d] - \int\limits_{\{x \in \Theta^d : x < \theta\}} x f(x) dx\right] \\
\frac{\partial u^d(\theta)}{\partial \theta} &= \frac{p}{2\mu(\Theta^d)}\left[\mathbb{P}[x < \theta, x \in \Theta^d] - \mathbb{P}[x > \theta, x \in \Theta^d]\right].
\end{aligned}
$$

If $\theta \leq \theta'$ for each $\theta' \in \Theta^d$, then

$$
\begin{aligned}
u^d(\theta) &= \frac{p}{2}\mathbb{P}[x > \theta | x \in \Theta^d] \left[\mathbb{E}[x | x \in \Theta^d] - \theta\right] \\
&= \frac{p}{2}\left[\int\limits_{\{x \in \Theta^d\}} \frac{x f(x) dx}{\mu(\Theta^d)} - \theta]\right] \\
\frac{\partial u^d(\theta)}{\partial \theta} &= -\frac{p}{2}.
\end{aligned}
$$

Finally, if $\theta \geq \theta'$ for each $\theta' \in \Theta^d$, then

$$
\begin{aligned}
u^d(\theta) &= \frac{p}{2}\mathbb{P}[x < \theta | x \in \Theta^d] \left[\theta - \mathbb{E}[x | x \in \Theta^d]\right] \\
&= \frac{p}{2}\left[\theta - \int\limits_{\{x \in \Theta^d\}} \frac{x f(x) dx}{\mu(\Theta^d)}]\right] \\
\frac{\partial u^d(\theta)}{\partial \theta} &= \frac{p}{2}.
\end{aligned}
$$

$\square$

E Segmentation in Coexistence Equilibrium

Proof of Theorem 3. We start with a simple observation: For the mechanism to make a positive profit, there has to be both agents who buy and sell at the marketplace. This means for a positive measure of agents, $u^m(\theta) \geq u^d(\theta)$ on both regions with $\frac{\partial u^m(\theta)}{\partial \theta} = 1$ and $\frac{\partial u^m(\theta)}{\partial \theta} = -1$.

We have shown in Appendix D that for an arbitrary segmentation of agents, the expected utility from search has a slope between -0.5 and 0.5.

Notice that if for an agent $\frac{\partial u^m(\theta)}{\partial \theta} = 1$, then for each $\theta' > \theta$, $q(\theta') = 1$ by the envelope condition and monotonicity of the allocation for an IC mechanism. Similarly, if $\frac{\partial u^m(\theta)}{\partial \theta} = -1$, then for each $\theta' < \theta$, $q(\theta') = -1$.

Given this, if for θ, $u^m(\theta) \geq u^d(\theta)$ and $\frac{\partial u^m(\theta)}{\partial \theta} = 1$, then for each $\theta' > \theta$, $u^m(\theta') \geq u^d(\theta')$ and similarly for the sellers. Thus, let $\underline{\theta}$ be the highest value such that $u^m(\theta) \geq u^d(\theta)$ and $q(\theta) = -1$ in the equilibrium. Similarly, let $\overline{\theta}$ be the lowest value such that $u^m(\theta) \geq u^d(\theta)$ and $q(\theta) = 1$.

There must be at least one type such that $u^m(\theta) = u^d(\theta)$. If not, either the utilities from search are above the utilities from the mechanism everywhere so that no one comes to the marketplace and the profit of the marketplace is zero or the utilities from the mechanism is strictly higher everywhere so everyone is in the mechanism and the mechanism can reduce the utilities until some IR constraint binds to strictly increase the profit.

Next, we argue that it cannot be the case that u^m and u^d are only tangent at $\underline{\theta}$ and $\overline{\theta}$, the utilities have to cross each other at these cutoffs: If they are only tangent but does not cross each other, then $\underline{\theta}$ or $\overline{\theta}$ would have to be the point of a kink on u^m. Then, only one of the cutoffs is at a kink, and an interval of agents near the other cutoff join the search market. Suppose there is a kink at $\overline{\theta}$. Then, if the search market is active, u^d has to be increasing at $\overline{\theta}$ since everyone above it is in the mechanism so that an agent with the value $\overline{\theta}$ can only be a buyer in the search market. For there to be agents in the search market, u^d should cross u^m at a point θ such that u^d is decreasing since it cannot cross u^m on the part it is constant or has a slope of 1 below $\overline{\theta}$. But if there is such a point, then u^d would be increasing at θ, since all agents in the search market will be below it as well, which shows this is impossible. Thus, u^m and u^d cannot be tangent at $\underline{\theta}$ and $\overline{\theta}$, they have to cross each other at these points.

Then, due to the shape of the feasible utility functions (u^m can have slopes -1, 0, and 1 in this order and u^d is first decreasing and then increasing -with a potentially constant 0 slope in the middle- with a slope that remains between $-\frac{1}{2}$ and $\frac{1}{2}$), either all agents with values in $[\underline{\theta}, \overline{\theta}]$ join the search market or the flat part of the u^m crosses u^d twice again, in which case agents with values in $[\underline{\theta}, a]$ and $[b, \overline{\theta}]$ join the search market for some $\underline{\theta} < a < b < \overline{\theta}$ and agents with values in a, b join the mechanism as well. Moreover, in the latter case, we need $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$; otherwise the u^d would be either strictly decreasing or strictly increasing for agents with values in $[a, b]$, in which case u^d and u^m would not cross at both a and b, as this case requires. When $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$, u^d would be flat, as it can be seen from the slope we computed above. So, we can write the profit as follows where the case with $a = b$ corresponds to the situation where the flat part of u^m does not cross u^d.

$$\Pi = -F(\underline{\theta})u^d(\underline{\theta}) - (1 - F(\overline{\theta}))u^d(\overline{\theta}) - (F(b) - F(a))u^d(a) - \int_0^{\underline{\theta}} C(x)f(x)dx + \int_{\overline{\theta}}^1 V(x)f(x)dx$$

Moreover, the constraints are $0 \leq \underline{\theta} \leq a \leq b \leq \overline{\theta} \leq 1$, $F(\underline{\theta}) \geq 1 - F(\overline{\theta})$ and $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$.

Let $\Theta^d = [\underline{\theta}, a] \cup [b, \overline{\theta}]$. Then, by noting that $(1 - F(\overline{\theta}) - F(\underline{\theta})) \leq 0$ by feasibility, it is easy to show that Π is decreasing in $\underline{\theta}$ in the feasible space. We will use this to show that the feasibility binds.

Next, we consider the Lagrangian problem to study the KKT conditions. Here, we will initially relax the problem by relaxing the equality constraint $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$ to $F(a) + F(b) \geq F(\underline{\theta}) + F(\overline{\theta})$ but focus on solutions where it binds. This is how we learn that the feasibility constraint must bind. We will then use feasibility to observe that $a = b$ should hold in the equilibrium, which means that any coexistence equilibrium has an interval of types in the decentralized market.

$$
\begin{aligned}
\mathcal{L}(\underline{\theta}, a, b, \overline{\theta}, \lambda) = \Pi &+ \lambda_1(F(\underline{\theta}) + F(\overline{\theta}) - 1) + \lambda_2(1 - \overline{\theta}) + \lambda_3(\overline{\theta} - b) + \lambda_4(b - a) + \lambda_5(a - \underline{\theta}) + \lambda_6\underline{\theta} \\
&+\lambda_7(F(a) + F(b) - F(\underline{\theta}) - F(\overline{\theta})) \\
&\frac{\partial \mathcal{L}}{\partial \underline{\theta}} = \frac{\partial \Pi}{\partial \underline{\theta}} + \lambda_1 f(\underline{\theta}) - \lambda_5 + \lambda_6 - \lambda_7 f(\underline{\theta}) = 0 \\
&\frac{\partial \mathcal{L}}{\partial a} = \frac{\partial \Pi}{\partial a} - \lambda_4 + \lambda_5 + \lambda_7 f(a) = 0 \\
&\frac{\partial \mathcal{L}}{\partial b} = \frac{\partial \Pi}{\partial b} - \lambda_3 + \lambda_4 + \lambda_7 f(b) = 0 \\
&\frac{\partial \mathcal{L}}{\partial \overline{\theta}} = \frac{\partial \Pi}{\partial \overline{\theta}} + \lambda_1 f(\overline{\theta}) - \lambda_3 + \lambda_4 - \lambda_7 f(\overline{\theta}) = 0 \\
&\lambda_i \geq 0 \\
&\lambda_1(F(\underline{\theta}) + F(\overline{\theta}) - 1) = 0 \\
&\lambda_2(1 - \overline{\theta}) = 0 \\
&\lambda_3(\overline{\theta} - b) = 0 \\
&\lambda_4(b - a) = 0 \\
&\lambda_5(a - \underline{\theta}) = 0 \\
&\lambda_6\underline{\theta} = 0 \\
&\lambda_7(F(a) + F(b) - F(\underline{\theta}) - F(\overline{\theta})) = 0.
\end{aligned}
$$

First, we note that for $\Pi > 0$, we need $1 > \overline{\theta}$. Moreover, for $1 > \overline{\theta}$, we need $\underline{\theta} > 0$ by feasibility. Then, we have $\lambda_2 = \lambda_6 = 0$ by complementary slackness conditions.

Remember that for $\underline{\theta} > 0$, we have $\frac{\partial \Pi}{\partial \underline{\theta}} < 0$. Then, since $\lambda_6 = 0$ and $\lambda_5, \lambda_7 \geq 0$, for $\frac{\partial \mathcal{L}}{\partial \underline{\theta}} = 0$, we need $\lambda_1 > 0$. By complementary slackness, this implies the feasibility constraint must bind.

Then, the profit function becomes:

$$
\begin{aligned}
\Pi = &- \frac{p(F(b) - F(a))}{2\left[F(\overline{\theta}) - F(b) + F(a) - F(\underline{\theta})\right]} \left[\int_b^{\overline{\theta}} x f(x) dx - \int_{\underline{\theta}}^{a} x f(x) dx\right] \\
&+ \frac{2-p}{2}\left[-\underline{\theta}F(\underline{\theta}) - \overline{\theta}F(\overline{\theta}) + \overline{\theta}\right].
\end{aligned}
$$

Next we are going to argue that in any solution to the above problem with a positive profit, we must have $a = b$. Suppose $(\underline{\theta}, a, b, \overline{\theta})$ maximizes Π and $b > a$. Remember that we reject any solution that does not satisfy $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$, since this is an equilibrium requirement. Then, the differences of integrals in the above equation is nonnegative. Moreover, it is strictly positive if $\Pi > 0$.

Notice that for $\Pi > 0$, we need $\overline{\theta} > \underline{\theta}$. If $\overline{\theta} = \underline{\theta}$, then it must be the case that $\overline{\theta} = a = b = \underline{\theta} = F^{-1}(0.5)$ and then we can verify that $\Pi = 0$. $\overline{\theta} > \underline{\theta}$ implies $\overline{\theta} > b$ and $a > \underline{\theta}$ because (i) we need $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$ in the equilibrium and (ii) $\overline{\theta} = b > a = \underline{\theta}$ cannot happen in the equilibrium as shown before stating the Lagrangian problem. But when we have $\overline{\theta} > b \geq a > \underline{\theta}$ and $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$, we have:

$$
\begin{aligned}
&\left[\int_b^{\overline{\theta}} xf(x)dx - \int_{\underline{\theta}}^{a} xf(x)dx\right] \\
&= \left[(F(\overline{\theta}) - F(b)) \int_b^{\overline{\theta}} \frac{xf(x)dx}{(F(\overline{\theta}) - F(b))} - (F(a) - F(\underline{\theta})) \int_{\underline{\theta}}^{a} \frac{xf(x)dx}{(F(a) - F(\underline{\theta}))}\right] \\
&= (F(\overline{\theta}) - F(b))\mathbb{E}[x|x \in [b, \overline{\theta}]] - (F(a) - F(\underline{\theta}))\mathbb{E}[x|x \in [\underline{\theta}, a]] \\
&= (F(\overline{\theta}) - F(b)) \left[\mathbb{E}[x|x \in [b, \overline{\theta}]] - \mathbb{E}[x|x \in [\underline{\theta}, a]]\right] > 0.
\end{aligned}
$$

Then, we must have $a = b$, since this does not effect the virtual surplus but minimizes the cost. Moreover, it must be the case that $F(a) = F(b) = F^{-1}(\frac{1}{2})$ since we need $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$ and the feasibility binds. Thus, any coexistence equilibrium must be such that in the equilibrium segmentation, an interval of intermediate types join the decentralized market. ☐

F Simplifying the Profit Function

Here we simplify the profit function in the general case.

$$
\Pi_{\underline{\theta}, \overline{\theta}} = \int_0^{\underline{\theta}} t(\theta)f(\theta)d\theta + \int_{\overline{\theta}}^{1} t(\theta)f(\theta)d\theta.
$$

We will study each integral separately. We start with the first one.

$$\int_{0}^{\underline{\theta}} t(\theta) f(\theta) d\theta = \int_{0}^{\underline{\theta}} \left[\theta q(\theta) - u^m(0)\right] f(\theta) d\theta - \int_{0}^{\underline{\theta}} \int_{0}^{\theta} q(x) f(\theta) dx d\theta \tag{1}$$

$$= \int_{0}^{\underline{\theta}} \left[\theta q(\theta) - u^m(0)\right] f(\theta) d\theta - \int_{0}^{\underline{\theta}} \int_{x}^{\underline{\theta}} q(x) f(\theta) d\theta dx \tag{2}$$

$$= \int_{0}^{\underline{\theta}} \left[\theta q(\theta) - u^m(0)\right] f(\theta) d\theta - \int_{0}^{\underline{\theta}} q(x) \int_{x}^{\underline{\theta}} f(\theta) d\theta dx \tag{3}$$

$$= \int_{0}^{\underline{\theta}} \left[\theta q(\theta) - u^m(0)\right] f(\theta) d\theta - \int_{0}^{\underline{\theta}} q(x) (F(\underline{\theta}) - F(x)) dx \tag{4}$$

$$= \int_{0}^{\underline{\theta}} \left[-u^m(0) + \left(x - \frac{F(\underline{\theta}) - F(x)}{f(x)}\right) q(x)\right] f(x) dx \tag{5}$$

$$= \int_{0}^{\underline{\theta}} \left[-u^m(\underline{\theta}) + \int_{0}^{\underline{\theta}} q(y) dy + \left(x - \frac{F(\underline{\theta}) - F(x)}{f(x)}\right) q(x)\right] f(x) dx \tag{6}$$

$$= \int_{0}^{\underline{\theta}} F(\underline{\theta}) q(y) \frac{f(y)}{f(y)} dy + \int_{0}^{\underline{\theta}} \left[-u^m(\underline{\theta}) + \left(x - \frac{F(\underline{\theta}) - F(x)}{f(x)}\right) q(x)\right] f(x) dx \tag{7}$$

$$= \int_{0}^{\underline{\theta}} \left[-u^m(\underline{\theta}) + \left(x + \frac{F(x)}{f(x)}\right) q(x)\right] f(x) dx = -F(\underline{\theta}) u^m(\underline{\theta}) + \int_{0}^{\underline{\theta}} \left[\left(x + \frac{F(x)}{f(x)}\right) q(x)\right] f(x) dx \tag{8}$$

In line 2, we change the order of integration; in line 3, we isolate the inner integral by extracting the allocations out; in line 4, we replace the value of the inner integral; in line 5, we merge the sum back; in line 6, we replace the value of the utility of the lowest type; in line 7, we integrate out the information rent for these types; in line 8, we cancel the new double integral with the $-\underline{\theta} q(x)$ as that integral turns out to be just the integral of $\underline{\theta} q(x)$ by changing the order of integration as above. We follow the similar steps for the transfers from $[\overline{\theta}, 1]$.

$$\int_{\overline{\theta}}^{1} t(\theta)f(\theta)d\theta = \int_{\overline{\theta}}^{1} \left[\theta q(\theta) - u^m(\overline{\theta}) - \int_{\overline{\theta}}^{\theta} q(x)dx\right] f(\theta)d\theta$$

$$= \int_{\overline{\theta}}^{1} \left[\theta q(\theta) - u^m(\overline{\theta})\right] f(\theta)d\theta - \int_{\overline{\theta}}^{1}\int_{\overline{\theta}}^{\theta} q(x)f(\theta)dxd\theta$$

$$= \int_{\overline{\theta}}^{1} \left[\theta q(\theta) - u^m(\overline{\theta})\right] f(\theta)d\theta - \int_{\overline{\theta}}^{1}\int_{x}^{1} q(x)f(\theta)d\theta dx$$

$$= \int_{\overline{\theta}}^{1} \left[\theta q(\theta) - u^m(\overline{\theta})\right] f(\theta)d\theta - \int_{\overline{\theta}}^{1} q(x)\int_{x}^{1} f(\theta)d\theta dx$$

$$= \int_{\overline{\theta}}^{1} \left[\theta q(\theta) - u^m(\overline{\theta})\right] f(\theta)d\theta - \int_{\overline{\theta}}^{1} q(x)(1 - F(x))dx$$

$$= \int_{\overline{\theta}}^{1} \left[-u^m(\overline{\theta}) + \left(x - \frac{1 - F(x)}{f(x)}\right) q(x)\right] f(x)dx$$

$$= -(1 - F(\overline{\theta}))u^m(\overline{\theta}) + \int_{\overline{\theta}}^{1} \left[\left(x - \frac{1 - F(x)}{f(x)}\right) q(x)\right] f(x)dx$$

G Profit from the coexistence equilibrium

Using the Lemma 2,

$$\int_{0}^{\underline{\theta}} C(x)q(x)f(x)dx + \int_{\overline{\theta}}^{1} V(y)q(y)f(y)dy = \int_{\overline{\theta}}^{1} V(y)f(y)dy - \int_{0}^{\underline{\theta}} C(x)f(x)dx$$

From the Appendix F, we have

$$-\int_{0}^{\underline{\theta}} C(x)f(x)dx + \int_{\overline{\theta}}^{1} V(y)f(y)dy = -\underline{\theta}F(\underline{\theta}) - \overline{\theta}F(\overline{\theta}) + \overline{\theta}$$

Moreover, under a CRS matching function M for the decentralized market, p is independent of the segmentation of the market. Then, using the binding IR constraints, the compensations paid to the agents will be given by:

$$
\begin{aligned}
&F(\underline{\theta})u^d(\underline{\theta}) + (1 - F(\overline{\theta}))u^d(\overline{\theta}) \\
&= p\frac{F(\underline{\theta})}{2}\left[\frac{\int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta) d\theta}{F(\overline{\theta}) - F(\underline{\theta})} - \underline{\theta}\right] + p\frac{1 - F(\overline{\theta})}{2}\left[\overline{\theta} - \frac{\int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta) d\theta}{F(\overline{\theta}) - F(\underline{\theta})}\right] \\
&= p\frac{1}{2}\left(F(\underline{\theta})\left[\frac{\int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta) d\theta}{F(\overline{\theta}) - F(\underline{\theta})} - \underline{\theta}\right] + (1 - F(\overline{\theta}))\left[\overline{\theta} - \frac{\int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta) d\theta}{F(\overline{\theta}) - F(\underline{\theta})}\right]\right) \\
&= \frac{p}{2}\left(F(\underline{\theta})E[\theta|\underline{\theta} \le \theta \le \overline{\theta}] - \underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta})) - (1 - F(\overline{\theta}))E[\theta|\underline{\theta} \le \theta \le \overline{\theta}]\right) \\
&= \frac{p}{2}\left(\left(F(\underline{\theta}) - (1 - F(\overline{\theta}))\right)E[\theta|\underline{\theta} \le \theta \le \overline{\theta}] - \underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta}))\right)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\Pi \\
&= -\frac{p}{2}\left(\left(F(\underline{\theta}) - (1 - F(\overline{\theta}))\right)E[\theta|\underline{\theta} \le \theta \le \overline{\theta}] - \underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta}))\right) + \left[-\underline{\theta}F(\underline{\theta}) - \overline{\theta}F(\overline{\theta}) + \overline{\theta}\right] \\
&= \frac{1}{2}\left((2 - p)\left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta}))\right] - p\left[F(\underline{\theta}) - (1 - F(\overline{\theta}))\right]E[\theta|\underline{\theta} \le \theta \le \overline{\theta}]\right)
\end{aligned}
$$

Notice that

$$\mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]] = \frac{\int_{\underline{\theta}}^{\overline{\theta}} x f(x) dx}{F(\overline{\theta}) - F(\underline{\theta})}$$

$$\frac{\partial \mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]]}{\partial \underline{\theta}} = \frac{-\underline{\theta} f(\underline{\theta})[F(\overline{\theta}) - F(\underline{\theta})] + f(\underline{\theta}) \left[\int_{\underline{\theta}}^{\overline{\theta}} x f(x) dx\right]}{[F(\overline{\theta}) - F(\underline{\theta})]^2}$$

$$= f(\underline{\theta}) \frac{\int_{\underline{\theta}}^{\overline{\theta}} x f(x) dx - \underline{\theta}[F(\overline{\theta}) - F(\underline{\theta})]}{[F(\overline{\theta}) - F(\underline{\theta})]^2}$$

$$= f(\underline{\theta}) \frac{\mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]] - \underline{\theta}}{[F(\overline{\theta}) - F(\underline{\theta})]} > 0.$$

Then, using this,

$$2\frac{\partial \Pi}{\partial \underline{\theta}}$$

$$= -(2-p)F(\underline{\theta}) - (2-p)\underline{\theta} f(\underline{\theta}) - p[f(\underline{\theta})]E[\theta|\underline{\theta} \leq \theta \leq \overline{\theta}] - p\left[F(\underline{\theta}) - (1 - F(\overline{\theta}))\right] \frac{\partial \mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]]}{\partial \underline{\theta}}$$

For a feasible mechanism, we need $F(\underline{\theta}) - (1 - F(\overline{\theta})) \geq 0$. Thus, each term above is negative, and Π is decreasing in $\underline{\theta}$. Given that Π is decreasing in $\underline{\theta}$, $F(\underline{\theta}) - (1 - F(\overline{\theta})) \geq 0$ will bind in any equilibrium. Therefore,

$$\Pi = \frac{1}{2}(2-p)\left[-\underline{\theta} F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta}))\right]$$

$$= \frac{2-p}{2}\left[-\underline{\theta} F(\underline{\theta}) + \overline{\theta}(F(\underline{\theta}))\right]$$

$$= \frac{2-p}{2}\left[F(\underline{\theta})[\overline{\theta} - \underline{\theta}]\right]$$

Notice that for interior values with $\overline{\theta} > \underline{\theta}$, $\Pi > 0$. Thus, positive profit is feasible and will be achieved in the equilibrium.

H Existence of a Coexistence Equilibrium

Proof of Theorem 1. In the Appendix G, I show that the profit from any pair of thresholds, $\underline{\theta}$ and $\overline{\theta}$, can be written as follows.

$$\Pi = \frac{1}{2}\left([2-p]\left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta}))\right] - p\left[F(\underline{\theta}) - (1-F(\overline{\theta}))\right] E[\theta | \underline{\theta} \leq \theta \leq \overline{\theta}]\right)$$

Our constraints are $0 \leq \underline{\theta} \leq \overline{\theta} \leq 1$ and $F(\underline{\theta}) \geq 1 - F(\overline{\theta})$. First, we cannot have $\overline{\theta} < m(F)$ where $m(F)$ is the median of F since that would require $0.5 > F(\overline{\theta}) \geq F(\underline{\theta}) \geq 1 - F(\overline{\theta}) \geq 0.5$. Second, Appendix G also shows that Π is strictly decreasing in $\underline{\theta}$. Then, $F(\underline{\theta}) \geq 1 - F(\overline{\theta})$ should bind. Third, $\underline{\theta} \leq m(F) \leq \overline{\theta}$ as a result of previous two observations. Thus, for a strictly increasing F, this is essentially a single parameter problem with a continuous objective and a compact domain. Hence, it has a solution by Weierstrass Theorem. Moreover, the solution is interior in the sense that the constraints $\underline{\theta} \leq m(F) \leq \overline{\theta}$ do not bind. If they did, then the profit would be 0 while it is possible to achieve a positive profit when they do not bind. (Appendix G shows this in more detail as well.) When the feasibility condition binds, the expectation term disappears from the profit. Then, the profit in this equilibrium is equal to the profit when there was no search market, times a constant, $\frac{2-p}{2}$. Thus, the solution must still have $\mathcal{C}(\underline{\theta}) = \mathcal{V}(\overline{\theta})$, as shown in the Proposition A.1. The mechanism is constructed so that no agent has any profitable deviation either in market choice or the message to the designer.

□

H.1 The Mechanism in the Coexistence Equilibrium

Proposition H.1. *In the coexistence equilibrium, the mechanism the designer offers has the following allocation and transfer rules:*

$$q(\theta)=\begin{cases}-1 & \textit{if } \theta \leq \dfrac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\underline{\theta}}{2}\\ 1 & \textit{if } \theta \geq \dfrac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\overline{\theta}}{2}\\ 0 & \textit{otherwise.}\end{cases}$$

$$t(\theta)=\begin{cases}-\dfrac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\underline{\theta}}{2} & \textit{if } \theta \leq \dfrac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\underline{\theta}}{2}\\ \dfrac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\overline{\theta}}{2} & \textit{if } \theta \geq \dfrac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\overline{\theta}}{2}\\ 0 & \textit{otherwise.}\end{cases}$$

Proof of Proposition H.1. First, we compute the transfers for the agents who join the mechanism using the binding IR constraints for $\underline{\theta}$ and $\overline{\theta}$.

In the coexistence equilibrium we construct, for an agent with $\theta \in [0, \underline{\theta}]$,

$$\begin{aligned}
t(\theta) &= \theta q(\theta) - u^m(0) - \int_0^{\theta} q(x)dx \\
&= \theta(-1) - u^m(0) - \theta(-1) \\
&= -u^m(0) = -u^d(\underline{\theta}) + \int_0^{\underline{\theta}} q(x)dx = -u^d(\underline{\theta}) + \int_0^{\underline{\theta}} (-1)dx \\
&= -\frac{p}{2}\left[E[x|x\in[\underline{\theta},\overline{\theta}]] - \underline{\theta}\right] - \underline{\theta} \\
&= -\frac{p}{2}E[x|x\in[\underline{\theta},\overline{\theta}]] - \underline{\theta}\frac{2-p}{2}
\end{aligned}$$

Similarly, we can compute the transfer of agents with $\theta \in [\overline{\theta}, 1]$,

$$\begin{aligned}
t(\theta) &= \theta q(\theta) - u^m(\overline{\theta}) - \int_{\overline{\theta}}^{\theta} q(x)dx \\
&= \theta(1) - u^m(\overline{\theta}) - 1(\theta - \overline{\theta}) \\
&= \overline{\theta} - u^m(\overline{\theta}) = \overline{\theta} - \frac{p}{2}\left[\overline{\theta} - E[x|x\in[\underline{\theta},\overline{\theta}]]\right] \\
&= \frac{p}{2}E[x|x\in[\underline{\theta},\overline{\theta}]] + \overline{\theta}\frac{2-p}{2}
\end{aligned}$$

Knowing these, the designer can offer -1 allocation to all agents whose valuations are below $\frac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\underline{\theta}}{2}$. Similarly, for agents with valuations above $\frac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\overline{\theta}}{2}$, 1 unit of allocation can be offered. In between, they are not offered any trade. This allocation is clearly increasing. Moreover, accompanied by the transfers $-\frac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\underline{\theta}}{2}$ for agents with negative allocations and $\frac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\overline{\theta}}{2}$ for agents with positive allocations, agents with values in $(\underline{\theta},\overline{\theta})$ would strictly prefer the search market. To see this, note that these agents have utilities with slopes -1 until u^m hits 0, then it is constant at 0 and then it has the slope 1, after $\frac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\overline{\theta}}{2}$. Moreover, u^m and u^d are equal at $\underline{\theta}$ and $\overline{\theta}$. Since the slope of u^d is bounded between $-\frac{p}{2}$ and $\frac{p}{2}$, and u^d is positive, u^m and u^d cannot cross each other at any point other than $\underline{\theta}$ and $\overline{\theta}$. Thus, u^d remains below u^m for values in $(\underline{\theta},\overline{\theta})$.

□

I Efficiency of Coexistence

I.1 Under Uniform Distribution

Proof of Proposition 6. Suppose everyone is in the search market. Then, the total welfare can be computed as follows:

$$\begin{aligned}\mathbb{E}[u^d(\theta)] &= p\int_0^1 [\theta\theta - (1-\theta)\theta]d\theta \\ &= p\int_0^1 [2\theta^2 - \theta]d\theta \\ &= p\left[\frac{2\theta^3}{3} - \frac{\theta^2}{2}\right]_0^1 = \frac{p}{6}.\end{aligned}$$

Next, we compute the welfare created by the marketplace alone in the coexistence equilibrium. The welfare marketplace generates will be more than enough to exceed the total welfare of the pure search market, so we do not need to compute the welfare created by the search market in the coexistence.

The profit function from the coexistence equilibrium under the uniform distribution is a constant times $\underline{\theta}(\overline{\theta}-\underline{\theta}) = \underline{\theta}(1-2\underline{\theta})$ using the fact that the feasibility binds so that $\underline{\theta} = 1-\overline{\theta}$. This is maximized at $\underline{\theta} = \frac{1}{4}$ and $\overline{\theta} = \frac{3}{4}$. Then, the welfare generated by the marketplace is

$$\int_{0.75}^1 \theta_b d\theta_b - \int_0^{0.25} \theta_s d\theta_s = \left[\frac{\theta^2}{2}\right]_{0.75}^1 - \left[\frac{\theta^2}{2}\right]_0^{0.25} = \frac{3}{16}.$$

The total welfare from the search market is $\frac{p}{6} \leq \frac{1}{6}$ for any matching function since the probability a meeting will be less than or equal to 1. Moreover, $\frac{3}{16} > \frac{1}{6}$. Thus, for any matching function, the coexistence equilibrium creates a welfare higher than the pure search market. □

I.2 Under General Distribution

Proof of Proposition 7. For the pure search market, the total welfare created is given by

$$
\begin{aligned}
&\int_0^1 \left[pF(\theta)\theta - p(1-F(\theta))\theta\right] f(\theta)d\theta \\
=&p\int_0^1 \theta\left[2F(\theta)-1\right] f(\theta)d\theta
\end{aligned}
$$

In the first line above, $pF(\theta)$ is the probability that the agent with the value θ meets with an agent with a value less than θ, so she gets θ in the trade and $p(1-F(\theta))$ is the probability that she meets with an agent whose value is higher so she loses θ. As with the uniform case, we ignore the transfers in the utilities as the transfers will cancel in the search market.

In the coexistence equilibrium, the total welfare created in the search market is

$$
\begin{aligned}
&\int_0^1 \left[p\left[\frac{F(\theta)-F(\underline{\theta})}{F(\overline{\theta})-F(\underline{\theta})}\right]\theta - p\left[\frac{F(\overline{\theta})-F(\theta)}{F(\overline{\theta})-F(\underline{\theta})}\right]\theta\right] f(\theta)d\theta \\
=&p\int_{\underline{\theta}}^{\overline{\theta}} \theta\left[\frac{2F(\theta)-F(\underline{\theta})-F(\overline{\theta})}{F(\overline{\theta})-F(\underline{\theta})}\right] f(\theta)d\theta \\
=&p\int_{\underline{\theta}}^{\overline{\theta}} \theta\left[\frac{2F(\theta)-1}{F(\overline{\theta})-F(\underline{\theta})}\right] f(\theta)d\theta
\end{aligned}
$$

The welfare generated by the marketplace in the coexistence is

$$
\int_{\overline{\theta}}^1 xf(x)dx - \int_0^{\underline{\theta}} xf(x)dx
$$

$$
\begin{aligned}
& p\int_{\underline{\theta}}^{\overline{\theta}} \theta\left[\frac{2F(\theta)-1}{F(\overline{\theta})-F(\underline{\theta})}\right] f(\theta)d\theta + \int_{\overline{\theta}}^{1} xf(x)dx - \int_{0}^{\underline{\theta}} xf(x)dx \geq p\int_{0}^{1}\theta\left[2F(\theta)-1\right]f(\theta)d\theta \\
\Longleftrightarrow\ & p\int_{\underline{\theta}}^{\overline{\theta}} \theta\left[\frac{2F(\theta)}{F(\overline{\theta})-F(\underline{\theta})}\right] f(\theta)d\theta + \int_{\overline{\theta}}^{1} xf(x)dx + p\int_{0}^{1}\theta f(\theta)d\theta \\
& \geq p\int_{\underline{\theta}}^{\overline{\theta}} \left[\frac{\theta f(\theta)d\theta}{F(\overline{\theta})-F(\underline{\theta})}\right] + p\int_{0}^{1}\theta\left[2F(\theta)\right]f(\theta)d\theta + \int_{0}^{\underline{\theta}} xf(x)dx \\
\Longleftrightarrow\ & p\int_{\underline{\theta}}^{\overline{\theta}} \theta\left[\frac{2F(\theta)}{F(\overline{\theta})-F(\underline{\theta})}\right] f(\theta)d\theta + (1+p)\int_{\overline{\theta}}^{1}\theta f(\theta)d\theta \\
& \geq p\left[\frac{2F(\underline{\theta})}{1-2F(\underline{\theta})}\right]\int_{\underline{\theta}}^{\overline{\theta}}\theta f(\theta)d\theta + 2p\int_{0}^{1}\theta\left[F(\theta)\right]f(\theta)d\theta + (1-p)\int_{0}^{\underline{\theta}} xf(x)dx
\end{aligned}
$$

In the first line above, either we have

$$
\int_{\underline{\theta}}^{\overline{\theta}} \theta\left[\frac{2F(\theta)-1}{F(\overline{\theta})-F(\underline{\theta})}\right] f(\theta)d\theta \geq \int_{0}^{1}\theta\left[2F(\theta)-1\right]f(\theta)d\theta
$$

in which case the inequality is satisfied for any p, since $\int_{\overline{\theta}}^{1} xf(x)dx - \int_{0}^{\underline{\theta}} xf(x)dx \geq 0$ when $F(\underline{\theta}) = 1 - F(\overline{\theta})$ or

$$
\int_{\underline{\theta}}^{\overline{\theta}} \theta\left[\frac{2F(\theta)-1}{F(\overline{\theta})-F(\underline{\theta})}\right] f(\theta)d\theta < \int_{0}^{1}\theta\left[2F(\theta)-1\right]f(\theta)d\theta.
$$

Thus, to show that for any p, the coexistence is more efficient, it is enough to show it with $p = 1$.

□

J Proof of Proposition 8

Proof of Proposition 8. If all agents join the centralized market, outside option for each agent is not trading. Thus, for this subgame, the designer should announce a mechanism where the individual rationality simply requires agents to get nonnegative payoffs. This is equivalent to the case of prohibitive search frictions where the probability of trade in the decentralized market is 0, so the analysis of the Appendix A applies. Then, we learn that the designer must exclude agents with intermediate values, $(\underline{\theta}, \overline{\theta})$ from trade. This means these agents get 0 payoffs in the monopoly equilibrium. So in this case, their payoff is 0 in both markets.

Moreover, by construction of Proposition H.1, the mechanism is always designed so that for each possible segmentation, the agents with intermediate valuations always receive a lower payoff than their expected payoff from search. Thus, for any segmentation other than the monopolization, they can expect a strictly higher payoff from the decentralized market.

Thus, joining the decentralized market weakly dominates joining the centralized market for agents with valuations in $(\underline{\theta}, \overline{\theta})$.

□

Proof of Corollary 9. Proof of the Proposition 8 above shows that for the intermediate types, joining the decentralized market is undominated. Then, what we need to argue to complete the proof of the corollary is that for the extreme types, joining the centralized market is undominated. However, this is evident as the centralized market gives them strictly higher payoffs in the coexistence equilibrium than their deviation payoff, i.e., their payoff from the decentralized trade. Thus, in the coexistence equilibrium, each agent plays an undominated strategy. □

K Extension with Double Auction

To compute all agents' expected payoffs from the decentralized market, we need to know the optimal bids of agents whose valuations lie outside $(\underline{\theta}, \overline{\theta})$. For agents with values below $\theta \leq \underline{\theta}$, if they join the decentralized market and get matched to someone, the best response is to bid $b(\underline{\theta})$ and for agents with values above $\overline{\theta}$, the best response is to bid $b(\overline{\theta})$. I show this in two steps. First, I show that for any agent, the optimal bid must lie in $[b(\underline{\theta}), b(\overline{\theta})]$. Then, I show that for agents with values less than $\underline{\theta}$ the optimal bid is $b(\underline{\theta})$ while for agents with values above $\overline{\theta}$, the optimal bid is $b(\overline{\theta})$. This is stated in the next lemma and proved in the Appendix K.1.

Lemma K.1. *For each* $\theta \in [0, \underline{\theta}]$, $b(\theta) = b(\underline{\theta})$ *and for each* $\theta \in [\overline{\theta}, 1]$, $b(\theta) = b(\overline{\theta})$.

Agents' utilities from the decentralized market:

Suppose $\theta \in [\underline{\theta}, \overline{\theta}]$. If the agent has the lower value, she has the lower bid, since the bidding function is monotone. Then, the agent gives up her endowment but gets paid. If the agent has the higher value, then, she has the higher bid so she gets the other's endowment and pays for it. Thus, expected payoff is

$$u^{da}(\theta) = pG(\theta)\left[\theta - \int_{\underline{\theta}}^{\theta} \frac{\frac{1}{2}[b(x)+b(\theta)]g(x)dx}{G(\theta)}\right] + p(1-G(\theta))\left[\int_{\theta}^{\overline{\theta}} \frac{\frac{1}{2}[b(x)+b(\theta)]g(x)dx}{1-G(\theta)} - \theta\right]$$

$$= p\theta[2G(\theta)-1] + \frac{p}{2}\int_{\theta}^{\overline{\theta}} [b(x)+b(\theta)]g(x)dx - \frac{p}{2}\int_{\underline{\theta}}^{\theta} [b(x)+b(\theta)]g(x)dx$$

$$= p\theta[2G(\theta)-1] + \frac{p}{2}(1-2G(\theta))b(\theta) + \frac{p}{2}\int_{\theta}^{\overline{\theta}} b(x)g(x)dx - \frac{p}{2}\int_{\underline{\theta}}^{\theta} b(x)g(x)dx$$

If $\theta \leq \underline{\theta}$, then, as above argument shows, the optimal bid is $b(\underline{\theta})$ and the expected payoff from the decentralized market is:

$$u^{da}(\theta) = p\left[\int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{2}[b(\underline{\theta})+b(x)]g(x)dx - \theta\right]$$

Similarly, agents with $\theta \geq \overline{\theta}$ bid $b(\overline{\theta})$ and get the expected payoff:

$$u^{da}(\theta) = p\left[\theta - \int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{2}[b(\overline{\theta})+b(x)]g(x)dx\right]$$

Next, we compute $u^{da}(\underline{\theta})$ and $u^{da}(\overline{\theta})$ by using the formulas for $b(\cdot)$, as the IR constraints of $\underline{\theta}$ and $\overline{\theta}$ will again play an important role in the equilibrium. The details can be found in the Appendix K.2 but here is the end result:

$$u^{da}(\underline{\theta}) = \frac{p}{2}\left[-\underline{\theta} + 4\int_{\underline{\theta}}^{G^{-1}(\frac{1}{2})} \left[G(x) - \frac{1}{2}\right]^2 dx + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]$$
$$u^{da}(\overline{\theta}) = \frac{p}{2}\left[\overline{\theta} + 4\int_{G^{-1}(\frac{1}{2})}^{\overline{\theta}} \left[G(x) - \frac{1}{2}\right]^2 dx - \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]$$

In the coexistence equilibrium, we are going to make the IR constraints of $\underline{\theta}$ and $\overline{\theta}$ bind, as otherwise decreasing the payment until they bind increases the profit. For these cutoffs to work, we need the slope of the utility from the decentralized market, $\frac{\partial u^{da}(\theta)}{\partial \theta}$ for types below

$\underline{\theta}$ to be greater than -1, since -1 is the allocation they will be offered in the mechanism and we want u^{da} to be less than u^m on this region. Moreover, we need the slope of u^{da} to be greater than -1 around $\underline{\theta}$ and the slope should be increasing (thus the utility function should be convex). Finally, we need the slope of the utility from the decentralized market to be less than 1 for agents with values above $\overline{\theta}$.

Lemma K.2.

$$\frac{\partial u^{da}(\theta)}{\partial \theta} = \begin{cases} -p & \textit{if } \theta \leq \underline{\theta} \\ p(G(\theta) - 1) & \textit{if } \underline{\theta} \leq \theta \leq \overline{\theta} \\ p & \textit{if } \theta \geq \overline{\theta} \end{cases}$$

Proof can be found in the Appendix K.3.

Since this is between -1 and 1, for agents with values in $[\underline{\theta}, \overline{\theta}]$, the designer can indeed offer lower utilities to these agents. One way of doing this would be offering the allocation -1 for agents between $\underline{\theta}$ and $G^{-1}(\frac{1}{2})$ and allocation 1 for agents between $G^{-1}(\frac{1}{2})$ and $\overline{\theta}$. This would make sure these agents are offered utilities lower than their expected payoff from the decentralized market since $u^m(\underline{\theta}) = u^{da}(\underline{\theta})$ and $u^m(\overline{\theta}) = u^{da}(\overline{\theta})$. (This may offer utilities below zero for some agents. The designer may not be concerned about this, since these agents are not wanted anyway. However, if the designer wishes the mechanism to offer nonnegative utilities, this can be achieved by flattening the utility when it reaches zero, as the Proposition H.1 does.)

Now, we look at the profit function. As before, it is equal to

$$\begin{aligned}
\Pi_{\underline{\theta},\overline{\theta}} &= \mathbb{P}[\theta \in [0, \underline{\theta}]]\mathbb{E}[t(\theta)|\theta \in [0, \underline{\theta}]] + \mathbb{P}[\theta \in [\overline{\theta}, 1]]\mathbb{E}[t(\theta)|\theta \in [\overline{\theta}, 1]] \\
&= \int_0^{\underline{\theta}} t(\theta)f(\theta)d\theta + \int_{\overline{\theta}}^1 t(\theta)f(\theta)d\theta \\
&= -F(\underline{\theta})u^d(\underline{\theta}) - (1 - F(\overline{\theta}))u^d(\overline{\theta}) + \int_0^{\underline{\theta}} \mathcal{C}(x)q(x)f(x)dx + \int_{\overline{\theta}}^1 \mathcal{V}(x)q(x)f(x)dx \\
&= -F(\underline{\theta})u^d(\underline{\theta}) - (1 - F(\overline{\theta}))u^d(\overline{\theta}) - \int_0^{\underline{\theta}} \mathcal{C}(x)f(x)dx + \int_{\overline{\theta}}^1 \mathcal{V}(x)f(x)dx
\end{aligned}$$

First two terms are the compensations for agents to join the centralized marketplace, while the last two terms are the total virtual surplus. We know expression for the total surplus from before because that part is unchanged:

$$-\int_0^{\underline{\theta}} \mathcal{C}(x)f(x)dx + \int_{\overline{\theta}}^1 \mathcal{V}(x)f(x)dx = -\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta})).$$

Next, we study the compensations, since now they will be different from what we had for the Nash bargaining.

$$\begin{aligned}
&F(\underline{\theta})u^d(\underline{\theta}) + (1 - F(\overline{\theta}))u^d(\overline{\theta}) \\
=&F(\underline{\theta})p\left[\frac{1}{2}\left[b(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right] - \underline{\theta}\right] + (1 - F(\overline{\theta}))p\left[\overline{\theta} - \frac{1}{2}\left[b(\overline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]\right] \\
=&\frac{p}{2}\left[F(\underline{\theta})\left[b(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx - 2\underline{\theta}\right] + (1 - F(\overline{\theta}))\left[2\overline{\theta} - b(\overline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]\right] \\
=&p\left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta}))\right] \\
+&\frac{p}{2}\left[\left[F(\underline{\theta}) + F(\overline{\theta}) - 1\right]\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx + F(\underline{\theta})b(\underline{\theta}) - (1 - F(\overline{\theta}))b(\overline{\theta})\right]
\end{aligned}$$

Remember that the optimal bidding strategy is given by

$$b(\theta) = \theta - \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2}$$

with $G(\theta) = \dfrac{F(x) - F(\underline{\theta})}{F(\overline{\theta}) - F(\underline{\theta})}$ on $[\underline{\theta}, \overline{\theta}]$ and $G^{-1}(\frac{1}{2}) = F^{-1}\left(\frac{F(\underline{\theta})+F(\overline{\theta})}{2}\right)$. Although it is relatively easy to show that a coexistence equilibrium exists with arbitrary distributions, the general solution to the profit maximization problem is too complicated to provide some useful comparative statics. Hence, I focus on the uniform distribution, $U[0, 1]$ from here on to be able to find a closed form solution to the problem above.

Using the uniform distribution, with some algebra (see Appendix K.4), we can show that

$$b(\theta) = \frac{\overline{\theta} + \underline{\theta} + 4\theta}{6},$$

$$\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx = \frac{\overline{\theta} + \underline{\theta}}{2}.$$

Now, we can plug these back into the expression for the profit to see that it is decreasing in $\underline{\theta}$. Thus, the feasibility must bind, which means $\underline{\theta} = 1 - \overline{\theta}$. Using this, we simplify the profit further and obtain the following simple expression for the profit. (The derivations can be followed in Appendix K.5.)

$$\Pi_{\underline{\theta},\overline{\theta}} = \frac{6-5p}{6}\left[\underline{\theta}(\overline{\theta} - \underline{\theta})\right] = \frac{6-5p}{6}\Pi^M$$

Clearly, this problem has an interior solution, which is the same as the solution of the problem of the marketplace when it operated on its own: The profits in two cases are equal up to a constant multiplier. Thus, almost everything we have seen under the Nash bargaining hold here with the uniform distribution, with the exception of the change ratio of the profit.

K.1 Optimal Bids for Agents in the Marketplace

Proof of Lemma K.1. Step 1: Bidding below $b(\underline{\theta})$ can never be optimal: For any bid $b \leq b(\underline{\theta})$, the agent sells her endowment with certainty but get paid less than she would get if she bid $b(\underline{\theta})$. Mathematically, the expected utility of an agent who bids $b \leq b(\underline{\theta})$ is given by:

$$\frac{1}{2}\int_{\underline{\theta}}^{\overline{\theta}} [b + b(x)]g(x)dx - \theta$$

Thus, bids strictly below $b(\underline{\theta})$ cannot be optimal.

Similarly, bids strictly above $b(\overline{\theta})$ cannot be optimal either. In that case, the agent's expected payoff would be

$$\theta - \frac{1}{2}\int_{\underline{\theta}}^{\overline{\theta}} [b + b(x)]g(x)dx$$

Hence, for each $\theta \in [0, \underline{\theta}] \cup [\overline{\theta}, 1]$, the optimal bid must be the bid of some type joins the decentralized market, that is: $b(\theta) \in [b(\underline{\theta}), b(\overline{\theta})]$.

Step 2: Let us first define the following notation: If an agent bids $b(\theta')$, then the expected price for selling is $p_s(\theta') = \frac{1}{2} \int\limits_{\underline{\theta}}^{\theta'} \frac{[b(\theta')+b(x)]g(x)dx}{1-G(\theta')}$ and the expected price for buying is $p_b(\theta') = \frac{1}{2} \int\limits_{\theta'}^{\overline{\theta}} \frac{[b(\theta')+b(x)]g(x)dx}{G(\theta')}$.

Since the best response of agent with value $\underline{\theta}$ is $b(\underline{\theta})$, her expected payoff from this bid should be higher than any other $b(\theta')$ by revealed preference. Then,

$$\begin{aligned} p_s(\underline{\theta}) - \underline{\theta} &\geq (1 - G(\theta')) \left[p_s(\theta') - \underline{\theta}\right] + G(\theta') \left[\underline{\theta} - p_b(\theta')\right] \\ &= \underline{\theta}[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\ \iff p_s(\underline{\theta}) - (1 - G(\theta'))p_s(\theta') + G(\theta')p_b(\theta') &\geq 2\underline{\theta}G(\theta') \end{aligned}$$

Suppose $\theta \leq \underline{\theta}$. We want to show that bidding $b(\underline{\theta})$ gives a higher payoff than any other type's bid $b(\theta')$:

$$\begin{aligned} p_s(\underline{\theta}) - \theta &\geq (1 - G(\theta')) \left[p_s(\theta') - \theta\right] + G(\theta') \left[\underline{\theta} - p_b(\theta')\right] \\ &= \theta[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\ p_s(\underline{\theta}) - (1 - G(\theta'))p_s(\theta') + G(\theta')p_b(\theta') &\geq 2\theta G(\theta') \end{aligned}$$

But this is true since $p_s(\underline{\theta}) - (1 - G(\theta'))p_s(\theta') + G(\theta')p_b(\theta') \geq 2\underline{\theta}G(\theta') \geq 2\theta G(\theta')$ where the first inequality follows from the revealed preference argument above and the second one follows from $\underline{\theta} \geq \theta$ and $G(\theta') \geq 0$.

Similarly, the best response of an agent with value $\underline{\theta}$ is $b(\overline{\theta})$. Thus,

$$\begin{aligned} \overline{\theta} - p_b(\overline{\theta}) &\geq (1 - G(\theta')) \left[p_s(\theta') - \overline{\theta}\right] + G(\theta') \left[\overline{\theta} - p_b(\theta')\right] \\ &= \overline{\theta}[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\ \iff 2\overline{\theta}(1 - G(\theta')) &\geq p_b(\overline{\theta}) + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \end{aligned}$$

Suppose $\theta \geq \overline{\theta}$. In this case, we want to show that bidding $b(\overline{\theta})$ gives a higher payoff than any other type's bid $b(\theta')$:

$$
\begin{aligned}
\theta - p_b(\overline{\theta}) &\geq (1 - G(\theta'))\,[p_s(\theta') - \theta] + G(\theta')\,[\theta - p_b(\theta')] \\
&= \theta[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\
\iff 2\theta(1 - G(\theta')) &\geq p_b(\overline{\theta}) + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta')
\end{aligned}
$$

Again, this is true since $2\theta(1 - G(\theta')) \geq 2\overline{\theta}(1 - G(\theta')) \geq p_b(\overline{\theta}) + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta')$ where the first inequality again follows from the revealed preference argument above and the second one follows from $\theta \geq \overline{\theta}$ and $1 - G(\theta') \geq 0$. □

K.2 Binding IR constraints

$$
\begin{aligned}
u^{da}(\underline{\theta}) &= p\left[\int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{2}[b(\underline{\theta}) + b(x)]g(x)dx - \underline{\theta}\right] \\
&= p\left[\frac{1}{2}\int_{\underline{\theta}}^{\overline{\theta}} [b(\underline{\theta}) + b(x)]g(x)dx - \underline{\theta}\right] \\
&= p\left[\frac{1}{2}\left[b(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right] - \underline{\theta}\right] \\
&= p\left[\frac{1}{2}\left[\underline{\theta} - \frac{\int_{G^{-1}(\frac{1}{2})}^{\underline{\theta}} \left[G(x) - \frac{1}{2}\right]^2 dx}{\left[G(\underline{\theta}) - \frac{1}{2}\right]^2} + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right] - \underline{\theta}\right] \\
&= \frac{p}{2}\left[-\underline{\theta} + 4\int_{\underline{\theta}}^{G^{-1}(\frac{1}{2})} \left[G(x) - \frac{1}{2}\right]^2 dx + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]
\end{aligned}
$$

Similarly, we obtain,

$$
\begin{aligned}
u^{da}(\overline{\theta}) &= p\left[\overline{\theta} - \int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{2}[b(\overline{\theta}) + b(x)]g(x)dx\right] \\
&= p\left[\overline{\theta} - \frac{1}{2}\left[b(\overline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]\right] \\
&= p\left[\overline{\theta} - \frac{1}{2}\left[\overline{\theta} - \frac{\int_{G^{-1}(\frac{1}{2})}^{\overline{\theta}} \left[G(x) - \frac{1}{2}\right]^2 dx}{\left[G(\overline{\theta}) - \frac{1}{2}\right]^2} + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]\right] \\
&= \frac{p}{2}\left[\overline{\theta} + 4\int_{G^{-1}(\frac{1}{2})}^{\overline{\theta}} \left[G(x) - \frac{1}{2}\right]^2 dx - \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]
\end{aligned}
$$

K.3 Slope of Utilities from the Double Auction

Proof of Lemma K.2. It is easy to verify that for agents with values less than $\underline{\theta}$, $\frac{\partial u^{da}(\theta)}{\partial \theta} = -p \geq -1$ and for agents with values above $\overline{\theta}$, $\frac{\partial u^{da}(\theta)}{\partial \theta} = p \leq 1$. Next we show that $\frac{\partial u^{da}(\theta)}{\partial \theta}$ is greater than -1 for $\underline{\theta}$ and less than 1 for $\overline{\theta}$.

First, we need the derivative of the bidding function:

$$b(\theta) = \theta - \frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2}$$

$$b'(\theta) = 1 - \frac{[G(\theta) - \frac{1}{2}]^4 - 2g(\theta)(G(\theta) - \frac{1}{2}) \int\limits_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^4}$$

$$= \frac{2g(\theta)(G(\theta) - \frac{1}{2}) \int\limits_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^4}$$

$$= 2g(\theta) \frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^3}$$

$$u^{da}(\theta) = p\theta[2G(\theta) - 1] + \frac{p}{2}(1 - 2G(\theta))b(\theta) + \frac{p}{2} \int\limits_{\theta}^{\overline{\theta}} b(x)g(x)dx - \frac{p}{2} \int\limits_{\underline{\theta}}^{\theta} b(x)g(x)dx$$

$$
\begin{aligned}
\frac{\partial u^{da}(\theta)}{\partial\theta} &= p\left[(2G(\theta)-1)+2\theta g(\theta)\right]+\frac{p}{2}\left[-2g(\theta)b(\theta)+(1-2G(\theta))b'(\theta)\right]-\frac{p}{2}2b(\theta)g(\theta)\\
&= p(2G(\theta)-1)+2pg(\theta)(\theta-b(\theta))+\frac{p}{2}\left[(1-2G(\theta))b'(\theta)\right]\\
&= p\left[G(\theta)-1+2g(\theta)(\theta-b(\theta))+\frac{1}{2}\left[(1-2G(\theta))b'(\theta)\right]\right]\\
&= p\left[G(\theta)-1+2g(\theta)\left[\theta-\theta+\frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta}[G(x)-\frac{1}{2}]^2dx}{[G(\theta)-\frac{1}{2}]^2}\right]+\frac{1}{2}(1-2G(\theta))b'(\theta)\right]\\
&= p\left[G(\theta)-1+2g(\theta)\frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta}[G(x)-\frac{1}{2}]^2dx}{[G(\theta)-\frac{1}{2}]^2}+\frac{1}{2}(1-2G(\theta))\left[2g(\theta)\frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta}[G(x)-\frac{1}{2}]^2dx}{[G(\theta)-\frac{1}{2}]^3}\right]\right]\\
&= p\left[G(\theta)-1+2g(\theta)\frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta}[G(x)-\frac{1}{2}]^2dx}{[G(\theta)-\frac{1}{2}]^2}-2g(\theta)\frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta}[G(x)-\frac{1}{2}]^2dx}{[G(\theta)-\frac{1}{2}]^2}\right]\\
&= p\left[G(\theta)-1\right]
\end{aligned}
$$

$\square$

K.4 Bids with Uniform Distribution

$$b(\theta) = \theta - \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2} = \theta - \frac{\int_{\frac{\underline{\theta}+\overline{\theta}}{2}}^{\theta} \left[\frac{x-\underline{\theta}}{\overline{\theta}-\underline{\theta}} - \frac{1}{2}\right]^2 dx}{\left[\frac{\theta-\underline{\theta}}{\overline{\theta}-\underline{\theta}} - \frac{1}{2}\right]^2}$$

$$= \theta - \frac{\int_{\frac{\underline{\theta}+\overline{\theta}}{2}}^{\theta} \left[\frac{2x-\underline{\theta}-\overline{\theta}}{2(\overline{\theta}-\underline{\theta})}\right]^2 dx}{\left[\frac{2\theta-\underline{\theta}-\overline{\theta}}{2(\overline{\theta}-\underline{\theta})}\right]^2} = \theta - \frac{\int_{\frac{\underline{\theta}+\overline{\theta}}{2}}^{\theta} \left[2x-\underline{\theta}-\overline{\theta}\right]^2 dx}{\left[2\theta-\underline{\theta}-\overline{\theta}\right]^2}$$

$$= \theta - \frac{\left[\frac{1}{2\times 3}\left[2x-\underline{\theta}-\overline{\theta}\right]^3\right]_{\frac{\underline{\theta}+\overline{\theta}}{2}}^{\theta}}{\left[2\theta-\underline{\theta}-\overline{\theta}\right]^2} = \theta + \frac{1}{6}\frac{\left[2\theta-\underline{\theta}-\overline{\theta}\right]^3}{\left[2\theta-\underline{\theta}-\overline{\theta}\right]^2}$$

$$= \theta - \frac{2\theta-\underline{\theta}-\overline{\theta}}{6} = \frac{4\theta+\underline{\theta}+\overline{\theta}}{6}$$

Next, we compute another expression from the profit function:

$$\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx = \int_{\underline{\theta}}^{\overline{\theta}} \frac{4\theta+\underline{\theta}+\overline{\theta}}{6}\frac{1}{\overline{\theta}-\underline{\theta}}dx = \frac{1}{6(\overline{\theta}-\underline{\theta})}\int_{\underline{\theta}}^{\overline{\theta}} (4\theta+\underline{\theta}+\overline{\theta})dx$$

$$= \frac{1}{6(\overline{\theta}-\underline{\theta})}\left[2\overline{\theta}^2 - 2\underline{\theta}^2 + (\overline{\theta}+\underline{\theta})(\overline{\theta}-\underline{\theta})\right] = \frac{1}{6(\overline{\theta}-\underline{\theta})}\left[3(\overline{\theta}+\underline{\theta})(\overline{\theta}-\underline{\theta})\right] = \frac{\overline{\theta}+\underline{\theta}}{2}$$

K.5 Profit from Coexistence Equilibrium under Double Auction

$$
\begin{aligned}
\Pi_{\underline{\theta},\overline{\theta}} &= -\int_{0}^{\underline{\theta}} C(x)f(x)dx + \int_{\overline{\theta}}^{1} \mathcal{V}(x)f(x)dx - F(\underline{\theta})u^d(\underline{\theta}) - (1-F(\overline{\theta}))u^d(\overline{\theta}) \\
&= \left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta}))\right] - p\left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta}))\right] \\
&-\frac{p}{2}\left[\left[F(\underline{\theta}) + F(\overline{\theta}) - 1\right]\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx + F(\underline{\theta})b(\underline{\theta}) - (1-F(\overline{\theta}))b(\overline{\theta})\right] \\
&= (1-p)\left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta}))\right] \\
&-\frac{p}{2}\left[\left[F(\underline{\theta}) + F(\overline{\theta}) - 1\right]\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx + F(\underline{\theta})b(\underline{\theta}) - (1-F(\overline{\theta}))b(\overline{\theta})\right] \\
&= (1-p)\left[-\underline{\theta}^2 + \overline{\theta}(1-\overline{\theta})\right] \\
&-\frac{p}{2}\left[\left[\underline{\theta} + \overline{\theta} - 1\right]\frac{\overline{\theta}+\underline{\theta}}{2} + \underline{\theta}\frac{5\underline{\theta}+\overline{\theta}}{6} - (1-\overline{\theta})\frac{5\overline{\theta}+\underline{\theta}}{6}\right]
\end{aligned}
$$

The profit is decreasing in $\underline{\theta}$:

$$
\begin{aligned}
\frac{\partial \Pi_{\underline{\theta},\overline{\theta}}}{\partial \underline{\theta}} &= (1-p)\left[-2\underline{\theta}\right] - \frac{p}{2}\left[\frac{\overline{\theta}+\underline{\theta}}{2} + \frac{1}{2}\left[\underline{\theta} + \overline{\theta} - 1\right] + \frac{5\underline{\theta}+\overline{\theta}}{6} + \underline{\theta}\frac{5}{6} - (1-\overline{\theta})\frac{1}{6}\right] \\
&= (1-p)\left[-2\underline{\theta}\right] - \frac{p}{2}\left[\frac{2}{3}(4\underline{\theta} + 2\overline{\theta} - 1)\right] \leq 0
\end{aligned}
$$

Notice that the first summand is negative and inside the brackets of the second summand is positive since $\underline{\theta} \geq 1 - \overline{\theta}$ by feasibility. Thus, the profit is decreasing in $\underline{\theta}$. Hence, the feasibility binds and we have $\underline{\theta} = 1 - \overline{\theta}$, otherwise decreasing $\underline{\theta}$ until the feasibility binds strictly increases the profit. Then, we have

$$
\begin{aligned}
\Pi_{\underline{\theta},\overline{\theta}} &= (1-p)\left[-\underline{\theta}^2 + \overline{\theta}(1-\overline{\theta})\right] - \frac{p}{2}\left[\underline{\theta}\frac{5\underline{\theta}+\overline{\theta}}{6} - (1-\overline{\theta})\frac{5\overline{\theta}+\underline{\theta}}{6}\right] \\
&= (1-p)\left[-\underline{\theta}^2 + \overline{\theta}(1-\overline{\theta})\right] - \frac{p}{2}\left[\underline{\theta}\frac{5\underline{\theta}+\overline{\theta}-5\overline{\theta}-\underline{\theta}}{6}\right] \\
&= (1-p)\left[-\underline{\theta}^2 + (1-\underline{\theta})\underline{\theta}\right] - \frac{p}{2}\left[\underline{\theta}\frac{5\underline{\theta}+(1-\underline{\theta})-5(1-\underline{\theta})-\underline{\theta}}{6}\right] \\
&= (1-p)\left[\underline{\theta}(1-2\underline{\theta})\right] - \frac{p}{2}\left[\underline{\theta}\frac{8\underline{\theta}-4}{6}\right] = (1-p)\left[\underline{\theta}(1-2\underline{\theta})\right] + \frac{p}{6}\left[\underline{\theta}(1-2\underline{\theta})\right] \\
&\frac{6-5p}{6}\left[\underline{\theta}(1-2\underline{\theta})\right] = \frac{6-5p}{6}\Pi^M
\end{aligned}
$$

L Multiple Designers

Mechanism design problems with multiple designers are notoriously difficult to solve. The difficulty lies in the fact that when there are multiple designers, each one wants to condition her mechanism to the others' and none of them wants to post a mechanism that only depends on the agents' types. As a result, revelation principle does not hold in environments with multiple designers. Despite recent developments (Feng and Hartline, 2018), mechanism design without revelation principle is itself a notorious problem, even with a single designer.

One way the literature has dealt with this problem of infinite regress between competing mechanism is to explicitly assume that the designers each choose a direct mechanism. Although this is not without loss anymore, it at least provides a tractable way to study the problem. I will first follow this literature and then illustrate what else can be achieved with a non-direct mechanism.

Suppose there are n marketplace designers competing with each other for agents' participation. Each of them is a profit-maximizer. Suppose they are restricted to choose direct mechanisms. The timeline is as follows: First, designers announce their mechanisms simultaneously. Then, agents choose which marketplace to join simultaneously. Finally, trades in each marketplace realize simultaneously.

From the analysis of the Appendix A, we know that this is equivalent to saying that each designer i chooses a pair of prices (bid-ask prices) p_b^i and p_s^i for buying and selling in the marketplace i, respectively.

This environment is essentially a model of Bertrand competition between the marketplaces where agents' roles as buyers and sellers are endogenously determined. The reasoning from the Bertrand Equilibrium still holds: If some designer i makes a positive profit, then another designer j can offer slightly smaller bid-ask spread, and increase her own profit discontinuously by recruiting the agents who were trading on the marketplace i. Thus, there cannot be an equilibrium where a designer makes a positive profit. Conversely, there cannot be an equilibrium where a designer makes a negative profit: It is always feasible to make 0 profit by setting a prohibitively high price to everyone.

Next, I argue that there are equilibria where all designers make 0 profit: Consider the strategy profiles where (i) at least one of the marketplaces set $p_b = p_s = m(F)$ where $m(F)$ is the median of the distribution of the valuations, (ii) the rest of the marketplaces either shut down or set prohibitively unfavorable prices, and (iii) all agents uniformly randomize over marketplaces where $p_b = p_s = m(F)$. Clearly, none of the designers or agents can improve their payoffs strictly and all such strategy profiles constitute equilibria.[1] Finally, there is no equilibrium where all marketplaces are inactive: If this happened, then each marketplace would simply deviate to announcing the baseline mechanism and make a positive profit.

Notice that $m(F)$ is the price that would clear the market if we had a competitive market: Setting the price at $m(F)$ ensures that the supply is equal to demand. Thus, the equilibria of this game replicate the Walrasian equilibrium due to the nature of the competition here.

We have seen that introducing another profit-maximizing marketplace reduced the profits to zero. In the previous section, we have observed that when we instead considered a competing decentralized market, the profit only decreased by a constant ratio, always less than the half. This stark difference is a result of the restriction to direct mechanisms when they are not without loss of generality. Next, I will briefly discuss the simplest way in which mechanisms can depend on each other and how it allows the designers to recoup the profits.

Now suppose the designers can announce any kind of mechanism. Thus, they can make their prices depend on the other marketplaces. One of the simplest ways to utilize this interdependence is by posting prices together with price-matching guarantees.

In an equilibrium with price-matching guarantees, each designer i announces prices p_b^i and p_s^i, and also promises to honor the prices $\min_j\{p_b^j : j = 1, \dots, n\}$ and $\max_j\{p_s^j : j = 1, \dots, n\}$ as well; so each agent can get the lowest price for buying and the highest price for selling from each marketplace. This is in line with introducing price-matching guarantees in a Bertrand competition setting and obtaining an equilibrium with monopoly pricing (Hess and Gerstner, 1991).

In the framework of this paper, it is an equilibrium for all designers to announce the prices that are equivalent to the baseline mechanism and for all agents to uniformly randomize over

[1] Notice that there are many other payoff-equivalent equilibria. For instance, for $n = 2$, suppose we have $p_b = p_s = m(F)$ on both marketplaces. Then, the agents are indifferent between the marketplaces. Any segmentation such that the market is cleared in each marketplace is also an equilibrium.

all the marketplaces. In this case, the designers are splitting the baseline profit equally. Thus, they essentially operate as a cartel.

It is worth noting that this is not the unique equilibrium in this environment. Specifically, it is still an equilibrium for each designer to announce the median of the valuations as the posted price for both the buyers and sellers.[2]

Finally, in addition to choosing among these marketplaces, suppose agents also have the option to join the decentralized market, after observing the posted mechanisms. Then, the same argument works with a minor modification: The designers would each post prices equivalent to the coexistence mechanism with price-matching guarantees and the agents with extreme values would randomize over the marketplaces uniformly. Since no marketplace has any incentives to deviate, they collectively act as the centralized market from the main model. Given that the centralized market is the same (on the equilibrium-path), no agent has any incentives to deviate either. Thus, the same segmentation would constitute an equilibrium with the addition that the agents with extreme values now randomize over the marketplaces.

References

Feng, Y. and Hartline, J. D. (2018). An end-to-end argument in mechanism design (prior-independent auctions for budgeted agents). In *2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 404–415. IEEE.

Hess, J. D. and Gerstner, E. (1991). Price-matching policies: An empirical case. *Managerial and Decision Economics*, 12(4):305–315.

Idem, B. (2021). Optimal marketplace design. *Working Paper*.

[2]Requiring coalition-proofness for the designers as well would make sure that they are collectively achieving the baseline profit in the equilibrium.